İSTANBUL TECHNICAL UNIVERSITY ★ INSTITUTE OF SCIENCE AND TECHNOLOGY

# NEUTRON STARS IN ALTERNATIVE THEORIES OF GRAVITY

**M.Sc. Thesis by**
**Vildan KELEŞ**

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| **Department** | : | **PHYSICS** |
| **Programme** | : | **PHYSICS** |

**JUNE 2011**

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**Date of Submission : 6 May 2011**
**Date of Examin : 7 June 2011**

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**JUNE 2011**

İSTANBUL TEKNİK ÜNİVERSİTESİ ★ FEN BİLİMLERİ ENSTİTÜSÜ

# ALTERNATİF GRAVİTASYON MODELLERİNDE NÖTRON YILDIZLARI

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| **Tezin Enstitüye Verildiği Tarih** | : | **6 Mayıs 2011** |
| **Tezin Savunulduğu Tarih** | : | **7 Haziran 2011** |

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**HAZİRAN 2011**

**FOREWORD**

I would like to express my gratitude to my thesis supervisors Assoc. Prof. Kazım Yavuz Ekşi and Prof. Dr. Cemsinan Deliduman for offering invaluable help in all possible ways, continuous encouragement and helpful critics throughout this research. Without their willingness to help and the motivation they had instilled in me, it would not be possible to complete this work.

Finally, I give my thanks to my precious husband Gökhan Tuğyanoğlu for being the meaning of my life and showing patience during my research.

June 2011 Vildan KELEŞ
B.S.

**TABLE OF CONTENTS**

**Page**

## ABBREVIATIONS

| | | |
|---|---|---|
| **EH** | : | Einstein-Hilbert |
| **EoM** | : | Equation of Motion |
| **EoS** | : | Equation of State |
| **GR** | : | General Relativity |
| **M-R** | : | Mass-Radius |
| **NS** | : | Neutron Star |

## LIST OF FIGURES

**Page**

## LIST OF SYMBOLS

| | | |
|---|---|---|
| $g$ | : | Determinant of the metric tensor |
| $I$ | : | Identity matrix |
| $P$ | : | Pressure |
| $R$ | : | Ricci scalar |
| $G_{\mu\nu}$ | : | Einstein tensor |
| $g_{\mu\nu}$ | : | Metric tensor |
| $g^{\mu\nu}$ | : | Inverse of the metric tensor |
| $R_{\mu\nu}$ | : | Ricci tensor |
| $R^{\alpha}{}_{\lambda\mu\nu}$ | : | Riemann tensor |
| $T_{\mu\nu}$ | : | Energy-Momentum tensor |
| $\rho$ | : | Energy density |
| $\delta$ | : | Variation |
| $\partial$ | : | Partial derivative |
| $\Gamma^{\rho}{}_{\nu\mu}$ | : | Levi-Civita connection |
| $\nabla$ | : | Covariant derivative |
| $\Box$ | : | D'Alembertian |

**NEUTRON STARS IN ALTERNATIVE THEORIES OF GRAVITY**


## SUMMARY

Einstein's general relativity is a theory of gravity that has successfully passed all the Solar System tests. General relativity is favored compared to its alternatives, because it is the simplest. As there are no differences in their predictions for tests in the weak gravitational field of the Solar System, it is important to compare general relativity and its alternatives in their predictions for tests that can be done in strong gravity regime.

Although black holes present the strongest gravitational fields in Nature, they do not help in the discrimination of gravity theories, since the vacuum solutions in alternative theories are the same as the ones in general relativity. Neutron stars, which come right after black holes in their gravitational strength, are the most suitable objects for comparing the predictions of general relativity and its alternatives in the strong gravity regime.

In this thesis, hydrostatic equilibrium equations for neutron stars are obtained, via a perturbative approach in a string theory motivated gravitation model. The mass-radius relations are obtained, for a variety of equations of state, by solving the structure equations of the star, and comparing with the observational results in the literature.

Comparison of the mass-radius relations obtained in the model with the observationally constrained mass-radius relation, the free parameter of the gravitation model, $\beta$, is constrained. According to our results, deviations from the observationally determined mass-radius relation and the known properties of neutron stars is prominent if the value of $\beta$ exceeds $10^{11}$ cm$^2$.

The maximum observed mass of a neutron star is about 2 solar masses. Some equations of state are not compatible with this observation, because they yield a maximum mass for a neutron star which is less than 2 solar masses. In this thesis, there are stable solution branches at high masses for also those equations of state and that they could be compatible with the observations.

**ALTERNATİF GRAVİTASYON MODELLERİNDE NÖTRON YILDIZLARI**

**ÖZET**

Einstein'ın genel görelilik kuramı Güneş Sistemi içinde yapılan tüm sınamalardan başarıyla geçmiş bir gravitasyon kuramıdır. Bu sınamalardan geçebilen alternatif gravitasyon kuramları ile karşılaştırıldığında genel görelilik basit olması dolayısıyla tercih edilmektedir. Güneş Sistemi'ndeki zayıf gravitasyonel alanda, genel görelilik ile alternatifleri arasında fark olmaması dolayısıyla bu kuramların yoğun gravitasyonel alanlardaki öngörülerinin karşılaştırılması önem kazanmaktadır.

Kara delikler doğadaki en yoğun gravitasyonel alanları sunmakla birlikte, alternatif kuramlardaki boşluk çözümlerinin genel görelilikteki çözümler ile aynı olması nedeniyle gravite kuramların birbirinden ayırdedilmesine olanak vermezler. Gravitasyonel alanlarının şiddeti bakımından kara deliklerden hemen sonra gelen nötron yıldızları genel görelilik ile alternatiflerinin öngörülerinin karşılaştırılması için en uygun nesnelerdir.

Bu tezde sicim kuramından güdülenen bir gravitasyon modelinde nötron yıldızı için hidrostatik denge denklemleri pertürbatif yöntemle elde edilmiştir. Yıldızın yapısı sayısal olarak çözülerek, farklı hal denklemleri için, kütle-yarıçap ilişkisi bulunmuş ve literatürdeki gözlemsel sonuçlar ile karşılaştırılmıştır.

Modelden elde edilen kütle-yarıçap ilişkisinin gözlemsel olarak belirlenmiş kütle-yarıçap ilişkisi ile karşılaştırılması sonucunda kuramdaki serbest parametre $\beta$ için kısıtlamalar elde edilmiştir. Sonuçlarımıza göre, $\beta$'nın değerinin $10^{11}$ $\mathrm{cm}^2$ mertebesinin üzerine çıkması durumunda gözlemle belirlenen kütle-yarıçap ilişkisinden ve bilinen nötron yıldızı özelliklerinden fazlaca uzaklaşılmaktadır.

Gözlenen en yüksek kütleli nötron yıldızı yaklaşık 2 Güneş kütlesindedir. Bazı hal denklemleri, genel görecelik çerçevesinde, nötron yıldızı için maksimum kütle olarak 2 Güneş kütlesinden daha küçük değerler öngördüklerinden bu gözlemle uyumlu değildirler. Bu tezde incelediğimiz gravitasyon modelinde, bazı hal denklemleri için büyük kütlelerde yeni türden kararlı çözümler olabileceği ve bu hal denklemlerinin gözlemlerle uyum sağlayabileceği görülmüştür.

# 1. INTRODUCTION

General Relativity explains gravity as a geometric property of space-time. Main successes of this theory are explanations it brought to phenomena such as precession of the perihelion of Mercury, bending of light near massive bodies and the gravitational redshift of light. However, all these tests are done inside the Solar System. Cosmologically, one of the aims of a theory of gravity is to explain the accelerating expansion of the universe.

In the recent studies, accelerating expansion of universe is inferred from data of Type Ia supernovae [3–5]. In general, for explaining the acceleration of cosmic expansion, two avenues are followed [6].

The first and the simplest idea is to add a cosmological constant to the action of general relativity. This constant can be thought to correspond to dark energy, and it can be computed using quantum field theory. However, computed value of the cosmological constant is $10^{120}$ times larger than the value indicated by the observations [7].

The second idea is to modify the theory of gravity to obtain acceleration. In the weak-field limit (e.g. Solar System tests) Einstein's General Relativity (GR) gives results highly consistent with the observations. However, there are alternatives to Einstein's General Relativity which have the same predictions in the weak-field regime. The difference between GR and alternatives might become prominent in the strong-gravity regime.

In Einstein's theory of general relativity the starting point is the Einstein-Hilbert action

$$I = \int d^4x\sqrt{-g}R. \tag{1.1}$$

Modifying Einstein-Hilbert Lagrangian in any way leads to a deformation in gravitational field dynamics at any length scale of interest. A very commonly adopted and seemingly simple idea explored in the recent literature is to replace the Ricci scalar, $R$, in the Einstein-Hilbert action with a function $f(R)$ of it. The $f(R)$ term must have a lower order expansion in Ricci scalar in order to include the general relativity as, perhaps, a weak-field limit of it.

This theory passes Solar System tests, brings an explanation to the late-time accelerated expansion of the universe and also works well in the strong-field regime. The action for this theory is

$$I = \int d^4x\sqrt{-g}f(R) \tag{1.2}$$

where $R$ is the Ricci scalar.

A related theory is described by Emilio Santos [8] as

$$I = \int d^4x\sqrt{-g}(R+F) \tag{1.3}$$

In Equation (**1.3**), the dimensions of terms inside $F$ are bound in the interval from $L^0$ to $L^{-4}$ and therefore $F$ can be written as

$$F = \Lambda + a_0 R + a_1 R^2 + a_2 R_{\mu\nu}R^{\mu\nu} + a_3 \Box R + a_4 \nabla^\mu\nabla^\nu R_{\mu\nu} + a_5 R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} \tag{1.4}$$

In this action $\Lambda$ is the cosmological constant but it is neglected. The $a_0$ is neglected again, because it only changes the coefficient of Ricci scalar in the Einstein Hilbert action. Other than that, term multiplied with $a_3$ has no contribution to the field equations. Therefore it can be neglected. We also know that the covariant derivative of the Einstein tensor is zero

$$\nabla^\nu(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R) = 0. \tag{1.5}$$

From the above equation,we obtain

$$\nabla^\mu\nabla^\nu R_{\mu\nu} = \frac{1}{2}\Box R. \tag{1.6}$$

Therefore, term multiplied with $a_4$ has no contribution to the field equations and it can be omitted. The Gauss-Bonnet term is

$$G = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} \tag{1.7}$$

which has no contribution to the field equations. Therefore, the contraction of Riemann tensors can be written in terms of square of Ricci scalar and contraction of Ricci tensors. Finally, the action (**1.3**) can be written as

$$I = \int d^4x\sqrt{-g}(R + \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu}) \tag{1.8}$$

where $\alpha$ and $\beta$ are free-parameters.

In an another study, Santos analyses effects of those parameters on the neutron stars' mass for some equations of state [9]. In that work he is more interested with the baryon number of the star, which is different than the approach taken in this thesis.

In this thesis, we adopt an alternative theory, in which the Einstein-Hilbert action is modified with one of the lowest possible order terms, which we take here as $R_{\mu\nu}R^{\mu\nu}$. There could also be terms such as $R^2$ and $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ in the same order, however we would like to analyze the modified gravity theory with only one parameter $\beta$ and see its effect on the mass-radius relation of the neutron stars. Therefore the action of modified gravity we are going the analyze is

$$I = \int d^4x\sqrt{-g}\left(R + \beta R_{\mu\nu}R^{\mu\nu}\right) \tag{1.9}$$

In the second chapter, we obtain the equations of motion of this theory by varying the action with respect to the metric tensor.

In the third chapter, all equations of motion are presented for spherically symmetric metric which has only diagonal components

$$g_{\mu\nu} = -e^{2\phi}dt^2 + e^{2\lambda}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{1.10}$$

To obtain and solve the hydrostatic equilibrium structure we use the perturbative method [10,11]. For the terms multiplied with $\beta$, the expressions derived from Einstein field equations are used. Then, from the 'tt' and 'rr' components of the field equations, we obtain the first and the second modified Tolman-Oppenheimer-Volkoff equations.

In Chapter 4, we solve the structure of neutron stars in this gravity model for 6 representative equations of state describing the dense matter of neutron stars. We present the mass-radius relation for $\beta$ changing in the range $-2\times10^{11}$ cm$^2$ to $2\times10^{11}$ cm$^2$. We identify that $\beta \sim 10^{11}$ produces results that can have observational consequences.

In the fifth chapter we discuss our results and conclude that recent observational constraints on the mass-radius relation requires that $|\beta| < 10^{12}$ cm$^2$.

## 2. EQUATIONS OF MOTION WITH THE VARIATIONAL METHOD

In this Chapter, we derive the equations of motion (EoM) by using the variational method. There are two different approaches to obtain the EoM: In the Palatini approach, the Levi-Civita connections are independent of the metric and both fields are varied to obtain EoM, and in the metric gravity approach, metric is the only independent field whereas Levi-Civita connection has the usual dependence to the metric. Therefore, only the metric is varied in the latter approach.

We start with defining action of our alternative theory. In the appropriate units, we have the geometric part of the action as

$$I = \int d^4x \sqrt{-g} \left( R + \beta R_{\mu\nu} R^{\mu\nu} \right) \quad \textbf{(2.1)}$$

We already know the variation of the matter part of the Lagrangian. It gives the energy-momentum tensor. Variation of the geometrical part is

$$\delta I = \int d^4x \left[ \delta\sqrt{-g} \left( R + \beta R_{\mu\nu} R^{\mu\nu} \right) + \sqrt{-g} \left( \delta R + \beta \delta R_{\mu\nu} R^{\mu\nu} + \beta R_{\mu\nu} \delta R^{\mu\nu} \right) \right] \quad \textbf{(2.2)}$$

Variations of scalars in the above equation can be written in terms of variation of tensors: Variation of metric's determinant is

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu}, \quad \textbf{(2.3)}$$

and the variation of the Ricci scalar is

$$\delta R = \delta g^{\mu\nu} R_{\mu\nu} + g^{\mu\nu} \delta R_{\mu\nu}. \quad \textbf{(2.4)}$$

We subsitute Equations (**2.3**) and (**2.4**) into Equation (**2.2**) to obtain

$$\begin{aligned} \delta I &= \int d^4x \sqrt{-g} \left( -\frac{1}{2} g_{\mu\nu} R - \frac{1}{2} \beta g_{\mu\nu} R_{ab} R^{ab} + R_{\mu\nu} \right) \delta g^{\mu\nu} \\ &\quad + \int d^4x \sqrt{-g} \left( g^{\mu\nu} \delta R_{\mu\nu} + \beta \delta R_{\mu\nu} R^{\mu\nu} + \beta R_{\mu\nu} \delta g^{a\mu} g^{b\nu} R_{ab} \right) \\ &\quad + \int d^4x \sqrt{-g} \left( \beta R_{\mu\nu} g^{a\mu} \delta g^{b\nu} R_{ab} + \beta R_{\mu\nu} g^{a\mu} g^{b\nu} \delta R_{ab} \right) \end{aligned} \quad \textbf{(2.5)}$$

Since the Ricci tensor and the inverse of the metric tensor are symmetric tensors, we can write

$$\delta I = \int d^4x\sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \frac{1}{2}\beta g_{\mu\nu}R_{ab}R^{ab} + 2\beta R_{\mu a}R_{\nu b}g^{ab}\right)\delta g^{\mu\nu} \quad \textbf{(2.6)}$$

$$+ \int d^4x\sqrt{-g}\left(g^{\mu\nu}\delta R_{\mu\nu} + 2\beta R^{\mu\nu}\delta R_{\mu\nu}\right) \quad \textbf{(2.7)}$$

We would like to take $\delta g^{\mu\nu}$ outside the parentheses. In the Equation (**2.6**) this is already the case. In the following, we are going to arrange the 1st and 2nd terms of Equation (**2.7**) to this desired form.

## 2.1 Calculation of $\int d^4x\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu}$

Here we calculate the 1st term of Equation (**2.7**). Ricci tensor is obtained as a contraction of Riemann tensor. Thus, the variation of the Riemann tensor could be used for obtaining the variation of the Ricci tensor, because of the simplicity of the former. The variation of the Riemann tensor can be written in terms of the covariant derivative of the variation of the Levi-Civita connection as

$$\delta R^{\rho}_{\mu\lambda\nu} = \nabla_\lambda\left(\delta\Gamma^{\rho}{}_{\nu\mu}\right) - \nabla_\nu\left(\delta\Gamma^{\rho}{}_{\lambda\mu}\right) \quad \textbf{(2.8)}$$

In the case of $\rho = \lambda$, the variation of the Ricci tensor will be

$$\delta R_{\mu\nu} = \left(\delta R^{\rho}{}_{\mu\rho\nu}\right) = \nabla_\rho\left(\delta\Gamma^{\rho}{}_{\nu\mu}\right) - \nabla_\nu\left(\delta\Gamma^{\rho}{}_{\rho\mu}\right) \quad \textbf{(2.9)}$$

in terms of the covariant derivatives of the variation of Levi-Civita connection. We can now plug these into the 1st term of the Equation (**2.7**)

$$\int d^4x\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu} = \int d^4x\sqrt{-g}g^{\mu\nu}\left[\nabla_\rho\left(\delta\Gamma^{\rho}{}_{\nu\mu}\right) - \nabla_\nu\left(\delta\Gamma^{\rho}{}_{\rho\mu}\right)\right] \quad \textbf{(2.10)}$$

As the covariant derivative of the metric vanishes, it is possible to take the covariant derivative out of the parentheses

$$\int d^4x\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu} = \int d^4x\sqrt{-g}\nabla_\sigma\left[g^{\mu\nu}\left(\delta\Gamma^{\sigma}{}_{\nu\mu}\right) - g^{\sigma\mu}\left(\delta\Gamma^{\rho}{}_{\rho\mu}\right)\right] \quad \textbf{(2.11)}$$

The variation of the Levi-Civita connections with respect to the metric are

$$\delta\Gamma^{\sigma}{}_{\nu\mu} = -\frac{1}{2}\left[g_{\lambda\nu}\nabla_\mu\left(\delta g^{\lambda\sigma}\right) + g_{\lambda\mu}\nabla_\nu\left(\delta g^{\lambda\sigma}\right) - g_{\nu\alpha}g_{\mu\beta}\nabla^\sigma\left(\delta g^{\alpha\beta}\right)\right] \quad \textbf{(2.12)}$$

and

$$\delta\Gamma^{\rho}{}_{\rho\mu} = -\frac{1}{2}\left[g_{\lambda\rho}\nabla_\mu\left(\delta g^{\lambda\rho}\right) + g_{\lambda\mu}\nabla_\rho\left(\delta g^{\lambda\rho}\right) - g_{\rho\alpha}g_{\mu\beta}\nabla^\rho\left(\delta g^{\alpha\beta}\right)\right]. \quad \textbf{(2.13)}$$

Substituting the last two equations into Equation (**2.11**) and using $g^{\mu\nu}g_{\lambda\nu} = \delta^{\mu}{}_{\lambda}$ we obtain

$$
\begin{aligned}
\int d^4x\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu} \;=\; & -\frac{1}{2}\int d^4x\sqrt{-g}\nabla_\sigma\delta^{\mu}{}_{\lambda}\nabla_\mu\left(\delta g^{\lambda\sigma}\right) \\
& -\frac{1}{2}\int d^4x\sqrt{-g}\nabla_\sigma\delta^{\nu}{}_{\lambda}\nabla_\nu\left(\delta g^{\lambda\sigma}\right) \\
& +\frac{1}{2}\int d^4x\sqrt{-g}\nabla_\sigma\delta^{\mu}{}_{\alpha}g_{\mu\beta}\nabla^\sigma\left(\delta g^{\alpha\beta}\right) \\
& +\frac{1}{2}\int d^4x\sqrt{-g}\nabla_\sigma g_{\lambda\rho}\nabla^\sigma\left(\delta g^{\lambda\rho}\right) \\
& +\frac{1}{2}\int d^4x\sqrt{-g}\nabla_\sigma\delta^{\sigma}{}_{\lambda}\nabla_\rho\left(\delta g^{\lambda\rho}\right) \\
& -\frac{1}{2}\int d^4x\sqrt{-g}\nabla_\sigma g_{\rho\alpha}\delta^{\sigma}{}_{\beta}\nabla^\rho\left(\delta g^{\alpha\beta}\right). \qquad \textbf{(2.14)}
\end{aligned}
$$

Renaming indices as $\alpha \to \lambda$ and $\rho \to \beta$, we get

$$
\begin{aligned}
\int d^4x\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu} \;=\; & \frac{1}{2}\int d^4x\nabla_\sigma\left[-\nabla_\mu\left(\delta g^{\mu\sigma}\right)-\nabla_\nu\left(\delta g^{\nu\sigma}\right)\right] \\
& +\frac{1}{2}\int d^4x\nabla_\sigma\left[g_{\alpha\beta}\nabla^\sigma\left(\delta g^{\alpha\beta}\right)+g_{\alpha\beta}\nabla^\sigma\left(\delta g^{\alpha\beta}\right)\right] \\
& +\frac{1}{2}\int d^4x\nabla_\sigma\left[\nabla_\rho\left(\delta g^{\sigma\rho}\right)-\nabla_\alpha\left(\delta g^{\alpha\sigma}\right)\right]. \qquad \textbf{(2.15)}
\end{aligned}
$$

In this equation, the 2nd term cancels the 5th term and the 3rd term cancels the 4th. Thus, the equation simplifies to

$$
\int d^4x\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu} = \int d^4x\sqrt{-g}\nabla_\sigma\left[g_{\alpha\beta}\nabla^\sigma\left(\delta g^{\alpha\beta}\right)-\nabla_\mu\left(\delta g^{\mu\sigma}\right)\right]. \qquad \textbf{(2.16)}
$$

As the covariant derivative of metric tensor is zero it can be taken inside the brackets:

$$
\begin{aligned}
\int d^4x\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu} \;&=\; \int d^4x\sqrt{-g}\nabla_\sigma\left[\nabla^\sigma\left(g_{\alpha\beta}\delta g^{\alpha\beta}\right)-\nabla_\mu\left(\delta g^{\mu\sigma}\right)\right] \\
&=\; \int d^4x\sqrt{-g}\nabla_\sigma\nabla^\sigma\left(g_{\alpha\beta}\delta g^{\alpha\beta}\right) \\
&\quad -\int d^4x\sqrt{-g}\nabla_\sigma\nabla_\mu\left(\delta g^{\mu\sigma}\right) \qquad \textbf{(2.17)}
\end{aligned}
$$

We will treat each integral on the right hand side separately. By defining

$$
T^\sigma = \nabla^\sigma\left(g_{\alpha\beta}\delta g^{\alpha\beta}\right) \qquad \textbf{(2.18)}
$$

the first integral becomes

$$
\int d^4x\sqrt{-g}\nabla_\sigma\nabla^\sigma\left(g_{\alpha\beta}\delta g^{\alpha\beta}\right) = \int d^4x\sqrt{-g}\nabla_\sigma T^\sigma \qquad \textbf{(2.19)}
$$

From the definition of the covariant derivative of rank-1 tensor, we can write

$$\begin{aligned}\int d^4x\sqrt{-g}\nabla_\sigma\nabla^\sigma\left(g_{\alpha\beta}\delta g^{\alpha\beta}\right) &= \int d^4x\sqrt{-g}\left(\partial_\sigma T^\sigma+\Gamma^\sigma{}_{\sigma\alpha}T^\alpha\right) \qquad (\mathbf{2.20})\\ &= \int d^4x\sqrt{-g}\partial_\sigma T^\sigma+\int d^4x\sqrt{-g}\Gamma^\sigma{}_{\sigma\alpha}T^\alpha.\end{aligned}$$

To calculate the first term on the right hand side, we integrate by parts,

$$\int d^4x\sqrt{-g}\left(\partial_\sigma T^\sigma\right)=\int d^4x\partial_\sigma\left(\sqrt{-g}T^\sigma\right)-\int d^4x\left(\partial_\sigma\sqrt{-g}\right)T^\sigma \qquad (\mathbf{2.21})$$

Since the variation at the boundary vanishes, the integral on the left vanishes and we obtain

$$\int d^4x\sqrt{-g}\left(\partial_\sigma T^\sigma\right)=-\int d^4x\left(\partial_\sigma\sqrt{-g}\right)T^\sigma \qquad (\mathbf{2.22})$$

Substituting this expression into equation (**2.20**) and using the expression

$$\partial_\lambda\sqrt{-g}=\sqrt{-g}\Gamma^\nu{}_{\nu\lambda} \qquad (\mathbf{2.23})$$

which is derived in Appendix C, we obtain

$$\int d^4x\sqrt{-g}\nabla_\sigma\nabla^\sigma\left(g_{\alpha\beta}\delta g^{\alpha\beta}\right)=\int d^4x\left[-\left(\partial_\sigma\sqrt{-g}\right)T^\sigma+\sqrt{-g}\Gamma^\sigma{}_{\sigma\alpha}T^\alpha\right] \qquad (\mathbf{2.24})$$

Renaming indices as $\sigma\to\nu$ and $\alpha\to\sigma$, we finally obtain

$$\begin{aligned}\int d^4x\sqrt{-g}\nabla_\sigma\nabla^\sigma\left(g_{\alpha\beta}\delta g^{\alpha\beta}\right) &= \int d^4x\left[-\sqrt{-g}\Gamma^\nu{}_{\nu\sigma}T^\sigma+\Gamma^\nu{}_{\nu\sigma}T^\sigma\sqrt{-g}\right]\\ &= 0 \qquad (\mathbf{2.25})\end{aligned}$$

The second part of variation of the Ricci tensor in Equation (**2.17**), $\int d^4x\sqrt{-g}\nabla_\sigma\nabla_\mu\left(\delta g^{\mu\sigma}\right)$, is

$$\int d^4x\sqrt{-g}\nabla_\sigma\nabla_\mu\left(\delta g^{\mu\sigma}\right)=\int d^4x\sqrt{-g}\nabla_\sigma A^\sigma, \qquad (\mathbf{2.26})$$

where we defined $A^\sigma=\nabla_\mu\left(\delta g^{\mu\sigma}\right)$. From the definition of covariant derivative of rank-1 tensor we infer that

$$\int d^4x\sqrt{-g}\nabla_\sigma A^\sigma=\int d^4x\sqrt{-g}\left(\partial_\sigma A^\sigma+\Gamma^\sigma{}_{\sigma\alpha}A^\alpha\right). \qquad (\mathbf{2.27})$$

Again, we can take $\sqrt{-g}$ inside to obtain $\left(\partial_\sigma A^\sigma\right)\sqrt{-g}=-\left(\partial_\sigma\sqrt{-g}\right)A^\sigma$ in the integral, and renaming indices as $\sigma\to\alpha$ and $\alpha\to\sigma$ gives

$$\int d^4x\sqrt{-g}\nabla_\sigma A^\sigma=\int d^4x\left[-\left(\partial_\sigma\sqrt{-g}\right)A^\sigma+\sqrt{-g}\Gamma^\alpha{}_{\alpha\sigma}A^\sigma\right] \qquad (\mathbf{2.28})$$

Now, again, using the expression

$$\partial_\sigma \sqrt{-g} = \sqrt{-g}\Gamma^\alpha{}_{\alpha\sigma} \qquad (2.29)$$

we obtain

$$\begin{aligned}\int d^4x\sqrt{-g}\nabla_\sigma A^\sigma &= \int d^4x\left(-\sqrt{-g}\Gamma^\alpha{}_{\alpha\sigma}A^\sigma + \sqrt{-g}\Gamma^\alpha{}_{\alpha\sigma}A^\sigma\right) \\ &= 0. \end{aligned} \qquad (2.30)$$

As a result, the first term in 2nd line in the Equation (**2.7**) vanishes:

$$\begin{aligned}\int d^4x\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu} &= \int d^4x\sqrt{-g}\nabla_\sigma\nabla^\sigma\left(g_{\alpha\beta}\delta g^{\alpha\beta}\right) - \int d^4x\sqrt{-g}\nabla_\sigma\nabla_\mu\left(\delta g^{\mu\sigma}\right) \\ &= 0 \end{aligned} \qquad (2.31)$$

## 2.2 Calculation of $2\beta\int d^4x\sqrt{-g}R^{\mu\nu}\delta R_{\mu\nu}$

Now we calculate the 2nd term of Equation (**2.7**). We start with using Equation (**2.9**)

$$2\beta\int d^4x\sqrt{-g}R^{\mu\nu}\delta R_{\mu\nu} = 2\beta\int d^4x\sqrt{-g}R^{\mu\nu}\left[\nabla_\rho\left(\delta\Gamma^\rho{}_{\nu\mu}\right) - \nabla_\nu\left(\delta\Gamma^\rho{}_{\rho\mu}\right)\right]. \qquad (2.32)$$

Substituting the covariant derivative of Levi-Civita connections and doing necessary arrangements and cancellations we obtain

$$\begin{aligned}2\beta\int d^4x\sqrt{-g}R^{\mu\nu}\delta R_{\mu\nu} &= -\int d^4x\sqrt{-g}\beta R^{\mu\nu}g_{\lambda\nu}\nabla_\rho\nabla_\mu\left(\delta g^{\lambda\rho}\right) \\ &\quad + \int d^4x\sqrt{-g}\beta R_{\mu\nu}\nabla_\rho\nabla^\rho\left(\delta g^{\mu\nu}\right) \\ &\quad + \int d^4x\sqrt{-g}\beta R^{\lambda\rho}g_{\mu\nu}\nabla_\rho\nabla_\lambda\left(\delta g^{\mu\nu}\right) \\ &\quad - \int d^4x\sqrt{-g}\beta R^{\mu\nu}g_{\lambda\mu}\nabla_\rho\nabla_\nu\left(\delta g^{\lambda\rho}\right) \end{aligned} \qquad (2.33)$$

Now we denote each term on the right hand side of the above equation with a latter,

$$\int d^4x\sqrt{-g}2\beta R^{\mu\nu}\delta R_{\mu\nu} = -A + B + C - D, \qquad (2.34)$$

where

$$A = \int d^4x\sqrt{-g}\beta R^{\mu\nu}g_{\lambda\nu}\nabla_\rho\nabla_\mu\left(\delta g^{\lambda\rho}\right), \qquad (2.35)$$

$$B = \int d^4x\sqrt{-g}\beta R_{\mu\nu}\nabla_\rho\nabla^\rho\left(\delta g^{\mu\nu}\right), \qquad (2.36)$$

$$C = \int d^4x\sqrt{-g}\beta R^{\lambda\rho}g_{\mu\nu}\nabla_\rho\nabla_\lambda\left(\delta g^{\mu\nu}\right), \qquad (2.37)$$

$$D = \int d^4x\sqrt{-g}\beta R^{\mu\nu}g_{\lambda\mu}\nabla_\rho\nabla_\nu\left(\delta g^{\lambda\rho}\right) \qquad (2.38)$$

We are going to compute each term separately.

The covariant derivative of the spherically symmetric metric is zero. Therefore, $A$ can be written as

$$A = \int d^4x\sqrt{-g}\beta R^{\mu\nu}\nabla_\rho\nabla_\mu\left(g_{\lambda\nu}\delta g^{\lambda\rho}\right). \quad \textbf{(2.39)}$$

In order to calculate the second covariant derivative, we define $A_{\mu\nu}{}^{\rho} = \nabla_\mu\left(g_{\lambda\nu}\delta g^{\lambda\rho}\right)$ and write the above equation as

$$A = \int d^4x\sqrt{-g}\beta R^{\mu\nu}\nabla_\rho A_{\mu\nu}{}^{\rho} \quad \textbf{(2.40)}$$

According to the definition of covariant derivative, $A$ could be written as

$$A = \beta\int d^4x\sqrt{-g}R^{\mu\nu}\left(\partial_\rho A_{\mu\nu}{}^{\rho} - \Gamma^{\alpha}{}_{\rho\mu}A_{\alpha\nu}{}^{\rho} - \Gamma^{\alpha}{}_{\rho\nu}A_{\mu\alpha}{}^{\rho} + \Gamma^{\rho}{}_{\rho\alpha}A_{\mu\nu}{}^{\alpha}\right) \quad \textbf{(2.41)}$$

In the case of the term $\sqrt{-g}R^{\mu\nu}(\partial_\rho A_{\mu\nu}{}^{\rho})$ we perform an integration by parts and then use the equation $\partial_\rho\sqrt{-g} = \sqrt{-g}\Gamma^{\alpha}{}_{\alpha\rho}$ to obtain a simplified expression

$$\begin{aligned} A &= \beta\int d^4x\left[-\left(\partial_\rho R^{\mu\nu}\right)\sqrt{-g}A_{\mu\nu}{}^{\rho} - \Gamma^{\mu}{}_{\rho\alpha}A_{\mu\nu}{}^{\rho}\sqrt{-g}R^{\alpha\nu} - \Gamma^{\nu}{}_{\rho\alpha}A_{\mu\nu}{}^{\rho}\sqrt{-g}R^{\mu\alpha}\right] \\ &= -\beta\int d^4x\sqrt{-g}\left[\left(\partial_\rho R^{\mu\nu}\right) + \Gamma^{\mu}{}_{\rho\alpha}R^{\alpha\nu} + \Gamma^{\nu}{}_{\rho\alpha}R^{\mu\alpha}\right]A_{\mu\nu}{}^{\rho}. \quad \textbf{(2.42)}\end{aligned}$$

In the above equation, the expression between the brackets is the covariant derivative of $R^{\mu\nu}$. Using also the definition of $A_{\mu\nu}{}^{\rho}$, this equation further simplifies to

$$A = -\beta\int d^4x\sqrt{-g}\nabla_\rho R^{\mu\nu}\nabla_\mu\left(g_{\lambda\nu}\delta g^{\lambda\rho}\right) \quad \textbf{(2.43)}$$

We now define $B_\rho{}^{\mu\nu} = \nabla_\rho R^{\mu\nu}$ and $T_\nu{}^{\rho} = g_{\lambda\nu}\delta g^{\lambda\rho}$ and write the above equation in terms of these tensors as

$$A = -\beta\int d^4x\sqrt{-g}B_\rho{}^{\mu\nu}\nabla_\mu\left(T_\nu{}^{\rho}\right). \quad \textbf{(2.44)}$$

Substituting expression for the covariant derivative of $T_\nu{}^{\rho}$ and using Equation (**C.7**) we obtain

$$A = \beta\int d^4x\sqrt{-g}\left[\Gamma^{\mu}{}_{\mu\alpha}B_\rho{}^{\alpha\nu} + \left(\partial_\mu B_\rho{}^{\mu\nu}\right) + \Gamma^{\nu}{}_{\mu\alpha}B_\rho{}^{\mu\alpha} - \Gamma^{\alpha}{}_{\mu\rho}B_\alpha{}^{\mu\nu}\right]T_\nu{}^{\rho}. \quad \textbf{(2.45)}$$

Terms inside the brackets constitute the covariant derivative of $B_\rho{}^{\mu\nu}$. Therefore equation (**2.45**) can be rewritten as

$$\begin{aligned} A &= \beta\int d^4x\sqrt{-g}\nabla_\mu B_\rho{}^{\mu\nu}T_\nu{}^{\rho} \\ &= \beta\int d^4x\sqrt{-g}\nabla_\mu\nabla_\rho R^{\mu\nu}g_{\lambda\nu}\delta g^{\lambda\rho} \\ &= \beta\int d^4x\sqrt{-g}\nabla_\alpha\nabla_\nu R^{\alpha\beta}g_{\mu\beta}\delta g^{\mu\nu}. \quad \textbf{(2.46)}\end{aligned}$$

Note that we renamed indices in the last line above.

Expression for $B$ (**2.36**) will be computed with the same methods. We make the definition

$$L^{\rho\mu\nu} = \nabla^{\rho}\left(\delta g^{\mu\nu}\right) \tag{2.47}$$

Then Equation (**2.36**) becomes

$$B = \int d^4x\sqrt{-g}\beta R_{\mu\nu}\nabla_{\rho}L^{\rho\mu\nu} \tag{2.48}$$

There will be 4 terms from the covariant derivative because $L^{\rho\mu\nu}$ is a rank-3 tensor:

$$B = \int d^4x\sqrt{-g}\beta R_{\mu\nu}\left(\partial_{\rho}L^{\rho\mu\nu} + \Gamma^{\rho}{}_{\rho\sigma}L^{\sigma\mu\nu} + \Gamma^{\mu}{}_{\rho\sigma}L^{\rho\sigma\nu} + \Gamma^{\nu}{}_{\rho\sigma}L^{\rho\mu\sigma}\right) \tag{2.49}$$

After following the same steps as in the calculation of $A$ we find a simplified expression for $B$:

$$B = -\beta\int d^4x\sqrt{-g}\left[\left(\partial_{\rho}R_{\mu\nu}\right) - \Gamma^{\sigma}{}_{\rho\mu}R_{\sigma\nu} - \Gamma^{\sigma}{}_{\rho\nu}R_{\mu\sigma}\right]L^{\rho\mu\nu}. \tag{2.50}$$

Terms inside the brackets constitute the covariant derivative of $R_{\mu\nu}$. Therefore $B$ is

$$\begin{aligned} B &= -\beta\int d^4x\sqrt{-g}\nabla_{\rho}R_{\mu\nu}L^{\rho\mu\nu} \\ &= -\beta\int d^4x\sqrt{-g}\nabla_{\rho}R_{\mu\nu}\nabla^{\rho}\left(\delta g^{\mu\nu}\right). \end{aligned} \tag{2.51}$$

We now define $Z_{\rho\mu\nu} = \nabla_{\rho}R_{\mu\nu}$ and write $B$ again as

$$B = -\beta\int d^4x\sqrt{-g}Z_{\rho\mu\nu}\nabla^{\rho}\left(\delta g^{\mu\nu}\right). \tag{2.52}$$

Using the definition of the covariant derivative and performing an integration by parts we obtain

$$\begin{aligned} B &= -\beta\int d^4x\left[-\left(\partial^{\rho}Z_{\rho\mu\nu}\right)\sqrt{-g}\delta g^{\mu\nu} - \left(\partial^{\rho}\sqrt{-g}\right)Z_{\rho\mu\nu}\delta g^{\mu\nu}\right] \\ &\quad -\beta\int d^4x\left[\sqrt{-g}\Gamma^{\rho\mu}{}_{\alpha}Z_{\rho\mu\nu}\delta g^{\alpha\nu} + \sqrt{-g}\Gamma^{\rho\nu}{}_{\alpha}Z_{\rho\mu\nu}\delta g^{\mu a}\right]. \end{aligned} \tag{2.53}$$

By using

$$-\partial^{\rho}\sqrt{-g} = \sqrt{-g}\Gamma^{\rho\sigma}{}_{\sigma} \tag{2.54}$$

$B$ can be written as

$$B = \beta\int d^4x\sqrt{-g}\left[\left(\partial^{\rho}Z_{\rho\mu\nu}\right) - \Gamma^{\alpha\rho}{}_{\rho}Z_{\alpha\mu\nu} - \Gamma^{\rho\alpha}{}_{\mu}Z_{\rho\alpha\nu} - \Gamma^{\rho\alpha}{}_{\nu}Z_{\rho\mu\alpha}\right]\delta g^{\mu\nu}. \tag{2.55}$$

Terms inside the brackets constitute the covariant derivative of $Z_{\rho\mu\nu}$. $B$ can be further simplified to

$$B = \beta \int d^4x\sqrt{-g}\nabla^{\rho} Z_{\rho\mu\nu}\delta g^{\mu\nu}. \qquad \textbf{(2.56)}$$

Substituting $Z_{\rho\mu\nu} = \nabla_{\rho} R_{\mu\nu}$ we finally obtain $B$ as

$$B = \beta \int d^4x\sqrt{-g}\nabla^{\rho}\nabla_{\rho} R_{\mu\nu}\delta g^{\mu\nu}. \qquad \textbf{(2.57)}$$

In the calculation of $C$ (**2.37**), we follow the same method used in the derivation of $A$ and $B$. As the covariant derivative of the metric is zero we can take $g_{\mu\nu}$ inside the brackets:

$$C = \int d^4x\sqrt{-g}\beta R^{\lambda\rho}\nabla_{\rho}\nabla_{\lambda}\left(g_{\mu\nu}\delta g^{\mu\nu}\right). \qquad \textbf{(2.58)}$$

Defining $D_{\lambda} = \nabla_{\lambda}\left(g_{\mu\nu}\delta g^{\mu\nu}\right)$ the above equation becomes

$$\begin{aligned} C &= \int d^4x\sqrt{-g}\beta R^{\lambda\rho}\nabla_{\rho} D_{\lambda} \\ &= \beta \int d^4x\sqrt{-g}R^{\lambda\rho}\left(\partial_{\rho} D_{\lambda} - \Gamma^{\alpha}{}_{\rho\lambda} D_{\alpha}\right) \\ &= -\beta \int d^4x\left[\left(\partial_{\rho} R^{\lambda\rho}\right)\sqrt{-g}D_{\lambda} + \left(\partial_{\rho}\sqrt{-g}\right)R^{\lambda\rho}D_{\lambda}\right] \\ &\quad -\beta \int d^4x\left[\sqrt{-g}\Gamma^{\alpha}{}_{\rho\lambda}R^{\lambda\rho}D_{\alpha}\right]. \end{aligned} \qquad \textbf{(2.59)}$$

Employing $\partial_{\rho}\sqrt{-g} = \sqrt{-g}\Gamma^{\alpha}{}_{\alpha\rho}$ again, we write

$$C = -\beta \int d^4x\sqrt{-g}\left[\left(\partial_{\rho} R^{\lambda\rho}\right) + \Gamma^{\rho}{}_{\rho\alpha}R^{\lambda\alpha} + \Gamma^{\lambda}{}_{\rho\alpha}R^{\alpha\rho}\right]D_{\lambda}. \qquad \textbf{(2.60)}$$

It can easily be seen that the term in the brackets is the covariant derivative of $R^{\lambda\rho}$. Therefore $C$ becomes,

$$\begin{aligned} C &= -\beta \int d^4x\sqrt{-g}\nabla_{\rho} R^{\lambda\rho}D_{\lambda} \\ &= -\beta \int d^4x\sqrt{-g}\nabla_{\rho} R^{\lambda\rho}\nabla_{\lambda}\left(g_{\mu\nu}\delta g^{\mu\nu}\right). \end{aligned} \qquad \textbf{(2.61)}$$

We now define $B^{\lambda} = \nabla_{\rho} R^{\lambda\rho}$ and $Y = g_{\mu\nu}\delta g^{\mu\nu}$

$$C = -\beta \int d^4x\sqrt{-g}B^{\lambda}\nabla_{\lambda} Y. \qquad \textbf{(2.62)}$$

We know that the covariant derivative of a scalar quantity equals to the partial derivative of that scalar. Using this

$$\begin{aligned} C &= -\beta \int d^4x\sqrt{-g}B^{\lambda}\left(\partial_{\lambda} Y\right) \\ &= -\beta \int d^4x\left[-\sqrt{-g}\left(\partial_{\lambda} B^{\lambda}\right)Y - \left(\partial_{\lambda}\sqrt{-g}\right)B^{\lambda}Y\right] \end{aligned} \qquad \textbf{(2.63)}$$

is obtained after integration by parts. Using the expression $\partial_\lambda\sqrt{-g} = \sqrt{-g}\Gamma^{\rho}{}_{\rho\lambda}$, $C$ becomes

$$C = \beta \int d^4x\sqrt{-g}\left[\left(\partial_\lambda B^\lambda\right) + \Gamma^{\lambda}{}_{\lambda\rho}B^\rho\right]Y. \tag{2.64}$$

Expression inside the brackets is the covariant derivative of $B^\lambda$. Therefore

$$C = \beta \int d^4x\sqrt{-g}\nabla_\lambda\nabla_\rho R^{\lambda\rho} g_{\mu\nu}\delta g^{\mu\nu}. \tag{2.65}$$

Calculation of $D$ is more involved and we present it in Appendix B. The result is

$$D = \beta \int d^4x\sqrt{-g}\nabla_\rho\nabla_\nu R^{\lambda\rho} g_{\mu\lambda}\delta g^{\mu\nu} \tag{2.66}$$

We now combine all of the terms $A, B, C$ and $D$ and write Equation (**2.34**) as

$$\begin{aligned}
\int d^4x\sqrt{-g}2\beta R^{\mu\nu}\delta R_{\mu\nu} \;=\; & -\beta\int d^4x\sqrt{-g}\nabla_\alpha\nabla_\nu R^{\alpha\beta}g_{\mu\beta}\delta g^{\mu\nu} \\
& +\beta\int d^4x\sqrt{-g}\nabla^\rho\nabla_\rho R_{\mu\nu}\delta g^{\mu\nu} \\
& +\beta\int d^4x\sqrt{-g}\nabla_\lambda\nabla_\rho R^{\lambda\rho}g_{\mu\nu}\delta g^{\mu\nu} \\
& -\beta\int d^4x\sqrt{-g}\nabla_\rho\nabla_\nu R^{\lambda\rho}g_{\mu\lambda}\delta g^{\mu\nu}
\end{aligned} \tag{2.67}$$

which is the 2nd term in Equation (**2.7**) brought to the desired form. Substituting this expression into the Equation (**2.7**) and remembering that the 1st term vanishes, we obtain

$$\begin{aligned}
\delta I \;=\; & \int d^4x\sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \frac{1}{2}\beta g_{\mu\nu}R_{ab}R^{ab}\right)\delta g^{\mu\nu} \\
& + \int d^4x\sqrt{-g}\left(2\beta R_{\mu a}R_{\nu b}g^{ab} - \beta\nabla_\alpha\nabla_\nu R^{\alpha\beta}g_{\mu\beta}\right)\delta g^{\mu\nu} \\
& + \int d^4x\sqrt{-g}\left(\beta\nabla^\rho\nabla_\rho R_{\mu\nu} + \beta\nabla_\lambda\nabla_\rho R^{\lambda\rho}g_{\mu\nu}\right)\delta g^{\mu\nu} \\
& - \int d^4x\sqrt{-g}\left(\beta\nabla_\rho\nabla_\nu R^{\lambda\rho}g_{\mu\lambda}\right)\delta g^{\mu\nu}
\end{aligned} \tag{2.68}$$

which can be rearranged to

$$\begin{aligned}
\delta I \;=\; & \int d^4x\sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\delta g^{\mu\nu} \\
& + \int d^4x\sqrt{-g}\beta\left(-\frac{1}{2}g_{\mu\nu}R_{ab}R^{ab} + 2R_{\mu a}R_{\nu b}g^{ab}\right)\delta g^{\mu\nu} \\
& + \int d^4x\sqrt{-g}\beta\left(-\nabla_\alpha\nabla_\nu R^{\alpha\beta}g_{\mu\beta} + \nabla^\rho\nabla_\rho R_{\mu\nu}\right)\delta g^{\mu\nu} \\
& + \int d^4x\sqrt{-g}\beta\left(\nabla_\lambda\nabla_\rho R^{\lambda\rho}g_{\mu\nu} - \nabla_\rho\nabla_\nu R^{\lambda\rho}g_{\mu\lambda}\right)\delta g^{\mu\nu}.
\end{aligned} \tag{2.69}$$

Combining result of this variation with the variation of the matter part we obtain the EoM as

$$\begin{aligned} 8\pi G T_{\mu\nu} &= G_{\mu\nu} + \beta\left(-\frac{1}{2}g_{\mu\nu}R_{ab}R^{ab} + 2R_{\mu a}R_{\nu b}g^{ab}\right) \\ &+\beta\left(-\nabla_\alpha\nabla_\nu R^{\alpha\beta}g_{\mu\beta} + \nabla^\rho\nabla_\rho R_{\mu\nu}\right) \\ &+\beta\left(\nabla_\lambda\nabla_\rho R^{\lambda\rho}g_{\mu\nu} - \nabla_\rho\nabla_\nu R^{\lambda\rho}g_{\mu\lambda}\right), \end{aligned} \quad \textbf{(2.70)}$$

where $T_{\mu\nu}$ is the energy-momentum tensor of the particular matter and

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \quad \textbf{(2.71)}$$

is the Einstein tensor.

In order to simplify this equation further, the following relations from [12] will be employed,

$$\nabla^\mu R_{\rho\mu} = \frac{1}{2}\nabla_\rho R \quad \textbf{(2.72)}$$

$$\nabla_\alpha\nabla_\nu R^\beta{}_\mu = \nabla_\nu\nabla_\alpha R^\beta{}_\mu + R^\beta{}_{\lambda\alpha\nu}R^\lambda{}_\mu - R^\lambda{}_{\mu\alpha\nu}R^\beta{}_\lambda \quad \textbf{(2.73)}$$

By using Equation (**2.73**), it is possible to change the places of the covariant derivatives in the field equation (**2.70**) in order to bring them to the form of $\nabla^\mu R_{\rho\mu}$ so that we can use the relation (**2.72**) :

$$\begin{aligned} \nabla_\alpha\nabla_\nu R^\alpha{}_\mu &= \nabla_\nu\nabla_\alpha R^\alpha{}_\mu + R^\alpha{}_{\lambda\alpha\nu}R^\lambda{}_\mu - R^\lambda{}_{\mu\alpha\nu}R^\alpha{}_\lambda \\ &= \nabla_\nu\nabla_\alpha R^\alpha{}_\mu + R_{\lambda\nu}R^\lambda{}_\mu - R_{\sigma\mu\alpha\nu}R^{\alpha\sigma} \end{aligned} \quad \textbf{(2.74)}$$

From the anti-symmetry property of the Riemann tensor ($R_{\sigma\mu\alpha\nu} = -R_{\sigma\mu\nu\alpha}$); we deduce that

$$\nabla_\alpha\nabla_\nu R^\alpha{}_\mu = \nabla_\nu\nabla_\alpha R^\alpha{}_\mu + R_{\lambda\nu}R^\lambda{}_\mu + R_{\sigma\mu\nu\alpha}R^{\alpha\sigma}. \quad \textbf{(2.75)}$$

Now we write $\nabla_\alpha R^\alpha{}_\mu$ as,

$$\nabla_\alpha R^\alpha{}_\mu = \nabla^\sigma R_{\sigma\mu} = \frac{1}{2}\nabla_\mu R \quad \textbf{(2.76)}$$

Substituting this into Equation (**2.75**), we obtain

$$\nabla_\alpha\nabla_\nu R^\alpha{}_\mu = \frac{1}{2}\nabla_\nu\nabla_\mu R + R_{\lambda\nu}R^\lambda{}_\mu + R_{\sigma\mu\nu\alpha}R^{\alpha\sigma}. \quad \textbf{(2.77)}$$

We use the same steps for the other terms in Equation (**2.70**):

$$\nabla_\lambda\nabla_\rho R^{\lambda\rho}g_{\mu\nu} = \nabla^a\nabla^b R_{ab}g_{\mu\nu} = \frac{1}{2}\nabla^a\nabla_a R g_{\mu\nu} = \frac{1}{2}\Box R g_{\mu\nu} \quad \textbf{(2.78)}$$

where we used the definition $\Box \equiv \nabla^a \nabla_a$.

We now use the tricks used in computation of Equation (**2.78**) for $\nabla_\rho \nabla_\nu R^\rho{}_\mu$:

$$\begin{aligned}
\nabla_\rho \nabla_\nu R^\rho{}_\mu &= \nabla_\nu \nabla_\rho R^\rho{}_\mu + R_{\lambda\nu} R^\lambda{}_\mu - R_{\sigma\mu\rho\nu} R^{\rho\sigma} \\
&= \nabla_\nu \nabla_\rho R^\rho{}_\mu + R_{\lambda\nu} R^\lambda{}_\mu + R_{\sigma\mu\nu\rho} R^{\rho\sigma} \\
&= \frac{1}{2} \nabla_\nu \nabla_\mu R + R_{\lambda\nu} R^\lambda{}_\mu + R_{\sigma\mu\nu\alpha} R^{\alpha\sigma}.
\end{aligned} \quad (\mathbf{2.79})$$

Using this definition in the equation of motion (**2.70**) we obtain

$$\begin{aligned}
8\pi G T_{\mu\nu} &= G_{\mu\nu} + \beta \left( -\frac{1}{2} g_{\mu\nu} R_{ab} R^{ab} + 2 R_{\mu a} R_{\nu b} g^{ab} \right) \\
&\quad -\beta \left( \frac{1}{2} \nabla_\nu \nabla_\mu R + R_{\lambda\nu} R^\lambda{}_\mu + R_{\sigma\mu\nu\alpha} R^{\alpha\sigma} \right) \\
&\quad +\beta \left( \nabla^\rho \nabla_\rho R_{\mu\nu} + \frac{1}{2} \Box R g_{\mu\nu} \right) \\
&\quad -\beta \left( \frac{1}{2} \nabla_\nu \nabla_\mu R + R_{\lambda\nu} R^\lambda{}_\mu + R_{\sigma\mu\nu\alpha} R^{\alpha\sigma} \right)
\end{aligned} \quad (\mathbf{2.80})$$

After a few arrangements and simplifications we arrive at the final result of this section

$$\begin{aligned}
8\pi G T_{\mu\nu} &= G_{\mu\nu} + \beta \left( -\frac{1}{2} g_{\mu\nu} R_{ab} R^{ab} + \nabla^\rho \nabla_\rho R_{\mu\nu} \right) \\
&\quad +\beta \left( -\nabla_\nu \nabla_\mu R - 2 R_{\sigma\mu\nu\alpha} R^{\alpha\sigma} + \frac{1}{2} \Box R g_{\mu\nu} \right).
\end{aligned} \quad (\mathbf{2.81})$$

This is the field equations of the alternative theory of gravity (**2.1**) whose neutron star solutions are analyzed in this thesis.

# 3. OBTAINING TOV EQUATIONS BY PERTURBATIVE METHOD

In this chapter we derive the hydrostatic equilibrium equation within the framework of the gravity model considered. The hydrostatic equilibrium equations, obtained and solved by Tolman-Oppenheimer and Volkoff [13, 14] within the framework of general relativity, are commonly called TOV equations. We, in this thesis, use the same nomenclature though the hydrostatic equilibrium equations in this gravity model will turn out to be quite different than the equations of Tolman-Oppenheimer and Volkoff.

## 3.1 Arranging the Equations of Motion by the Perturbative Method

In the previous chapter, we found that the equations of motion (EoM) are

$$\begin{aligned} 8\pi G T_{\mu\nu} &= G_{\mu\nu} + \beta\left(-\frac{1}{2}g_{\mu\nu}R_{ab}R^{ab} + \nabla^{\rho}\nabla_{\rho}R_{\mu\nu}\right) \\ &\quad -\beta\left(\nabla_{\nu}\nabla_{\mu}R + 2R_{\sigma\mu\nu\alpha}R^{\alpha\sigma} - \frac{1}{2}\Box R g_{\mu\nu}\right) \end{aligned} \tag{3.1}$$

These equations are going to be solved for the case of spherically symmetric metric. As in the case of general relativity we choose to work with a diagonal form of the metric and metric functions depend only on the radial coordinate $r$. In matrix form metric is

$$g_{\mu\nu} = \begin{bmatrix} -e^{2\phi} & 0 & 0 & 0 \\ 0 & e^{2\lambda} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{bmatrix}. \tag{3.2}$$

The energy-momentum tensor is the one of the perfect fluid, which in the rest frame of the fluid has the diagonal form

$$T^{\mu}{}_{\nu} = \begin{bmatrix} -\rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{bmatrix}. \tag{3.3}$$

By using $g_{\mu\nu}$, we can calculate all terms in the EoM which we denote as

$$M = R_{ab}R^{ab} \quad (3.4)$$

$$N = \nabla^{\rho}\nabla_{\rho}R_{\mu\nu} \quad (3.5)$$

$$S = \nabla_{\nu}\nabla_{\mu}R \quad (3.6)$$

$$F = R_{\sigma\mu\nu\alpha}R^{\alpha\sigma} \quad (3.7)$$

$$Y = \Box R \quad (3.8)$$

We start with the calculation of $M$:

$$\begin{aligned} M &= R_{ab}R^{ab} = R^{a}{}_{b}R^{a}{}_{b} \\ &= R^{0}{}_{0}R^{0}{}_{0} + R^{1}{}_{1}R^{1}{}_{1} + R^{2}{}_{2}R^{2}{}_{2} + R^{3}{}_{3}R^{3}{}_{3} \end{aligned} \quad (3.9)$$

The Einstein field equations in general relativity are

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = T_{\mu\nu}, \quad (3.10)$$

where we set $8\pi G = 1$. In order to compute Equation (**3.9**) we need the trace of Einstein's field equation which is

$$R = T + \frac{4}{2}R \quad (3.11)$$

or rather

$$R = -T \quad (3.12)$$

where

$$T = -\rho + 3P. \quad (3.13)$$

is the trace of the energy-momentum tensor. By plugging the result of Equation (**3.13**) into the Equation (**3.12**), we obtain the Ricci scalar

$$R = \rho - 3P \quad (3.14)$$

in terms of hydrodynamic quantities.

Multiplying both sides of Equation (**3.10**) with $g^{\mu\nu}$ we obtain

$$g^{\mu\nu}R_{\mu\nu} = g^{\mu\nu}T_{\mu\nu} + \frac{1}{2}Rg^{\mu\nu}g_{\mu\nu}. \quad (3.15)$$

Here $g^{\mu\nu}g_{\mu\nu}$ (no summations) equals to four-dimensional identity matrix $I$ and therefore

$$R^{\mu}{}_{\mu} = T^{\mu}{}_{\mu} + \frac{1}{2}R \cdot I. \quad \textbf{(3.16)}$$

In order to calculate the first term in Equation (**3.9**) we need to know $R^{0}{}_{0}$, which is found to be

$$\begin{aligned} R^{0}{}_{0} &= T^{0}{}_{0} + \frac{1}{2}R \cdot I \\ &= -\rho + \frac{1}{2}(\rho - 3P) \\ &= -\frac{1}{2}(\rho + 3P). \end{aligned} \quad \textbf{(3.17)}$$

By using the same method we obtain

$$\begin{aligned} R^{1}{}_{1} &= T^{1}{}_{1} + \frac{1}{2}R \cdot I \\ &= P + \frac{1}{2}(\rho - 3P) \\ &= \frac{1}{2}(\rho + 3P). \end{aligned} \quad \textbf{(3.18)}$$

Due to symmetries of energy-momentum tensor of a perfect fluid we infer that the other components of $R^{\mu}{}_{\nu}$ are

$$R^{1}{}_{1} = R^{2}{}_{2} = R^{3}{}_{3} = \frac{1}{2}(\rho + 3P). \quad \textbf{(3.19)}$$

By plugging these results into Equation (**3.9**), we obtain

$$\begin{aligned} R_{ab}R^{ab} &= \left[-\frac{1}{2}(\rho + 3P)\right]^2 + 3\left[\frac{1}{2}(\rho + 3P)\right]^2 \\ &= \rho^2 + 3P^2. \end{aligned} \quad \textbf{(3.20)}$$

The above term is common to all components of EoM. There is one more general term that is common to all components of EoM, $\Box R$. Since $R$ is a scalar we use the well-known formula

$$Y = \Box R = \frac{1}{\sqrt{g}}\partial_\rho\left(\sqrt{g}g^{\rho\sigma}\partial_\sigma R\right), \quad \textbf{(3.21)}$$

where $g$ denotes the determinant of the metric.

Metric has only diagonal components. Hence $\sigma$ equals to $\rho$. In addition to this $\rho$ must be "$r$", because $R$ depends only on the radial coordinate. For other values of $\sigma$ , partial derivative of $R$ will be zero. Therefore $\sigma = \rho = r$. In that case, Equation (**3.21**) can be easily written as

$$\begin{aligned} Y &= \frac{1}{\sqrt{g}}\partial_r\left(\sqrt{g}g^{rr}\partial_r R\right) \\ &= \frac{1}{\sqrt{g}}\partial_r\left(\sqrt{g}e^{-2\lambda}R'\right). \end{aligned} \tag{3.22}$$

Here, $R'$ denotes partial derivative of $R$ with respect to $r$. Applying the second $r$ derivative we find

$$\begin{aligned} Y &= \frac{1}{\sqrt{g}}\left(\frac{1}{2\sqrt{g}}g'e^{-2\lambda}R' - 2\lambda' e^{-2\lambda}\sqrt{g}R' + \sqrt{g}e^{-2\lambda}R''\right) \\ &= e^{-2\lambda}\left[R'' + \left(\frac{1}{2}\frac{g'}{g} - 2\lambda'\right)R'\right]. \end{aligned} \tag{3.23}$$

We now calculate $F$:

$$\begin{aligned} F &= R_{\sigma\mu\nu\alpha}R^{\alpha\sigma} = -R_{\mu\sigma\nu\alpha}R^{\alpha}{}_{\rho}g^{\rho\sigma} \\ &= -R^{\lambda}{}_{\sigma\nu\alpha}R^{\alpha}{}_{\rho}g^{\rho\sigma}g_{\mu\lambda} \end{aligned} \tag{3.24}$$

For $tt$ component, $\mu = \nu = t$ and $\mu$ must be equal to $\lambda$ due to metric being diagonal. For the same reason $\rho$ must be equal to $\sigma$. Therefore,

$$R_{\sigma tt\alpha}R^{\alpha\sigma} = -R^{t}{}_{\sigma t\alpha}R^{\alpha}{}_{\rho}g^{\rho\sigma}g_{tt} \tag{3.25}$$

is obtained.

Let us write all terms in the $tt$ component:

$$\begin{aligned} R_{\sigma tt\alpha}R^{\alpha\sigma} &= -R^{t}{}_{ttt}R^{t}{}_{t}g^{tt}g_{tt} - R^{t}{}_{rtr}R^{r}{}_{r}g^{rr}g_{tt} \\ &\quad -R^{t}{}_{\theta t\theta}R^{\theta}{}_{\theta}g^{\theta\theta}g_{tt} - R^{t}{}_{\phi t\phi}R^{\phi}{}_{\phi}g^{\phi\phi}g_{tt} \end{aligned} \tag{3.26}$$

If we plug in the whole Ricci tensor and Riemann tensor in the above equation

$$\begin{aligned} R_{\sigma tt\alpha}R^{\alpha\sigma} &= e^{2\phi}(e^{-2\phi})\frac{1}{2}(\rho + 3P)(0) \\ &\quad + e^{2\phi}e^{-2\lambda}\frac{1}{2}(\rho - P)\left[\phi'\lambda' - (\phi')^2 - \phi''\right] \\ &\quad + e^{2\phi}\frac{1}{2r^2}(\rho - P)\left(-re^{-2\lambda}\phi'\right) \\ &\quad + e^{2\phi}\frac{1}{2r^2}\sin^{-2}\theta(\rho - P)\left(-r\sin^2\theta e^{-2\lambda}\phi'\right). \end{aligned} \tag{3.27}$$

Arranging the above equation we find

$$R_{\sigma tt\alpha}R^{\alpha\sigma} = e^{2(\phi-\lambda)}\left[\frac{1}{2}\left(\phi'\lambda' - (\phi')^2 - \phi''\right) - \frac{1}{r}\phi'\right](\rho - P). \qquad \textbf{(3.28)}$$

For the $rr$ component of EoM, Equation (**3.24**) become

$$R_{\sigma rr\alpha}R^{\alpha\sigma} = -R^{r}{}_{\sigma r\alpha}R^{\alpha}{}_{\rho}g^{\rho\sigma}g_{rr} \qquad \textbf{(3.29)}$$

By writing all possibilities for Equation (**3.29**), we obtain

$$\begin{aligned} R_{\sigma rr\alpha}R^{\alpha\sigma} &= -R_{rtrt}R^{t}{}_{t}g^{tt} - R^{r}{}_{rrr}R^{r}{}_{r} \\ &\quad -R^{r}{}_{\theta r\theta}R^{\theta}{}_{\theta}g^{\theta\theta}g_{rr} - R^{r}{}_{\phi r\phi}R^{\phi}{}_{\phi}g^{\phi\phi}g_{rr} \end{aligned} \qquad \textbf{(3.30)}$$

$R^{r}{}_{rrr} = 0$ is zero due to symmetries of Riemann tensor. The above equation in terms of the hydrodynamic quantities then becomes

$$\begin{aligned} R_{\sigma rr\alpha}R^{\alpha\sigma} &= \frac{1}{2}(\rho + 3P)\left[\phi'\lambda' - (\phi')^2 - \phi''\right] \\ &\quad -e^{2\beta}\frac{1}{2r^2}(\rho - P)re^{-2\lambda}\lambda' \\ &\quad -e^{2\beta}\frac{1}{2r^2}(\rho - P)\sin^{-2}\theta re^{-2\lambda}\lambda'\sin^2\theta. \end{aligned} \qquad \textbf{(3.31)}$$

After a few arrangements, we can write the above equation as

$$\begin{aligned} R_{\sigma rr\alpha}R^{\alpha\sigma} &= \frac{1}{2}(\rho + 3P)\left[\phi'\lambda' - (\phi')^2 - \phi''\right] \\ &\quad -\frac{1}{r}\lambda'(\rho - P). \end{aligned} \qquad \textbf{(3.32)}$$

$\theta\theta$ and $\phi\phi$ components can be calculated similarly. Without giving the details, we only state the results:

$$\begin{aligned} R_{\sigma\theta\theta\alpha}R^{\alpha\sigma} &= re^{-2\lambda}\left[-\frac{1}{2}\phi'(\rho + 3P) - \frac{1}{2}\lambda'(\rho - P)\right] \\ &\quad -\frac{1}{2}\left(1 - e^{-2\lambda}\right)(\rho - P), \end{aligned} \qquad \textbf{(3.33)}$$

$$\begin{aligned} R_{\sigma\phi\phi\alpha}R^{\alpha\sigma} &= re^{-2\lambda}\left[-\frac{1}{2}\phi'(\rho + 3P) - \frac{1}{2}\lambda'(\rho - P)\right]\sin^2\theta \\ &\quad -\frac{1}{2}(\rho - P)\left(1 - e^{-2\lambda}\right)\sin^2\theta \end{aligned} \qquad \textbf{(3.34)}$$

To calculate $\nabla_\nu\nabla_\mu R$ we expand covariant derivatives

$$\begin{aligned} \nabla_\nu\nabla_\mu R &= \nabla_\nu\left(\partial_\mu R\right) \\ &= \partial_\nu\partial_\mu R - \Gamma^{\alpha}{}_{\nu\mu}\partial_\alpha R \end{aligned} \qquad \textbf{(3.35)}$$

First we calculate contribution of this term to $tt$ component of EoM. By replacing $\nu$ and $\mu$, both with $t$

$$\nabla_t \nabla_t R = \partial_t \partial_t R - \Gamma^{\alpha}{}_{tt} \partial_\alpha R. \tag{3.36}$$

Partial derivative of $R$ is different than zero only for derivatives with respect to $r$. Hence $\partial_t \partial_t R = 0$ , and only $\alpha = r$ gives nontrivial results. Therefore we obtain

$$\begin{aligned} \nabla_t \nabla_t R &= -\Gamma^{r}{}_{tt} \partial_r R \\ &= -e^{2(\phi - \lambda)} \phi' R'. \end{aligned} \tag{3.37}$$

Using a similar procedure, we calculate respective expressions for $rr$, $\theta\theta$ and $\phi\phi$ components:

$$\nabla_r \nabla_r R = \partial_r \partial_r R - \Gamma^{r}{}_{rr} \partial_r R \tag{3.38}$$

$$= R'' - \lambda' R', \tag{3.39}$$

$$\nabla_\theta \nabla_\theta R = \partial_\theta \partial_\theta R - \Gamma^{r}{}_{\theta\theta} \partial_r R \tag{3.40}$$

$$= r e^{-2\lambda} R', \tag{3.41}$$

$$\nabla_\phi \nabla_\phi R = \partial_\phi \partial_\phi R - \Gamma^{r}{}_{\phi\phi} \partial_r R \tag{3.42}$$

$$= r e^{-2\lambda} \sin^2 \theta R'. \tag{3.43}$$

The last and hardest part is the calculation of $\Box R_{\mu\nu}$. We are going to show steps only for $\Box R_{tt}$ and then only give results for the other three components. In the calculation we use Ricci tensor with one upper one lower component. Therefore,

$$\begin{aligned} \Box R_{\mu\nu} &= \nabla^\rho \nabla_\rho R_{\mu\nu} \\ &= \nabla^\rho \nabla_\rho R^{\sigma}{}_{\nu} g_{\sigma\mu} \\ &= g_{\sigma\mu} \nabla^\rho \nabla_\rho R^{\sigma}{}_{\nu} \end{aligned} \tag{3.44}$$

Using the definition of the covariant derivative for rank-2 tensor, we write

$$\begin{aligned} \Box R_{\mu\nu} &= g_{\sigma\mu} \nabla^\rho \left( \partial_\rho R^{\sigma}{}_{\nu} + \Gamma^{\sigma}{}_{\rho\alpha} R^{\alpha}{}_{\nu} - \Gamma^{\alpha}{}_{\rho\nu} R^{\sigma}{}_{\alpha} \right) \\ &= g_{\sigma\mu} g^{\rho\lambda} \nabla_\lambda \left( \partial_\rho R^{\sigma}{}_{\nu} + \Gamma^{\sigma}{}_{\rho\alpha} R^{\alpha}{}_{\nu} - \Gamma^{\alpha}{}_{\rho\nu} R^{\sigma}{}_{\alpha} \right). \end{aligned} \tag{3.45}$$

Taking the second covariant derivative gives us a very long expression for $\Box R_{\mu\nu}$:

$$
\begin{aligned}
\Box R_{\mu\nu} \;=\;& g_{\sigma\mu}g^{\rho\lambda}\left[\partial_\lambda\partial_\rho R^\sigma{}_\nu - \Gamma^\alpha{}_{\lambda\rho}\partial_\alpha R^\sigma{}_\nu + \Gamma^\sigma{}_{\lambda\alpha}\partial_\rho R^\alpha{}_\nu\right] \\
&+g_{\sigma\mu}g^{\rho\lambda}\left[-\Gamma^\alpha{}_{\lambda\nu}\partial_\rho R^\sigma{}_\alpha + \partial_\lambda\left(\Gamma^\sigma{}_{\rho\alpha}R^\alpha{}_\nu\right)\right] \\
&+g_{\sigma\mu}g^{\rho\lambda}\left[\Gamma^\sigma{}_{\lambda\beta}\Gamma^\beta{}_{\rho\alpha}R^\alpha{}_\nu - \Gamma^\beta{}_{\lambda\rho}\Gamma^\sigma{}_{\beta\alpha}R^\alpha{}_\nu\right] \\
&-g_{\sigma\mu}g^{\rho\lambda}\left[\Gamma^\beta{}_{\lambda\alpha}\Gamma^\sigma{}_{\rho\beta}R^\alpha{}_\nu\right] \\
&+g_{\sigma\mu}g^{\rho\lambda}\left[\Gamma^\alpha{}_{\lambda\beta}\Gamma^\sigma{}_{\rho\alpha}R^\beta{}_\nu - \Gamma^\beta{}_{\lambda\nu}\Gamma^\sigma{}_{\rho\alpha}R^\alpha{}_\beta\right] \\
&-g_{\sigma\mu}g^{\rho\lambda}\left[\partial_\lambda\left(\Gamma^\alpha{}_{\rho\nu}R^\sigma{}_\alpha\right) + \Gamma^\alpha{}_{\lambda\beta}\Gamma^\beta{}_{\rho\nu}R^\sigma{}_\alpha\right] \\
&+g_{\sigma\mu}g^{\rho\lambda}\left[\Gamma^\beta{}_{\lambda\rho}\Gamma^\alpha{}_{\beta\nu}R^\sigma{}_\alpha + \Gamma^\beta{}_{\lambda\nu}\Gamma^\alpha{}_{\rho\beta}R^\sigma{}_\alpha\right] \\
&+g_{\sigma\mu}g^{\rho\lambda}\left[-\Gamma^\sigma{}_{\lambda\beta}\Gamma^\alpha{}_{\rho\nu}R^\beta{}_\alpha + \Gamma^\beta{}_{\lambda\alpha}\Gamma^\alpha{}_{\rho\nu}R^\sigma{}_\beta\right]. \qquad \textbf{(3.46)}
\end{aligned}
$$

In order to compute the above equation, we need to know values of Levi-Civita connections for spherical-symmetric metric. The calculations of these symbols can be found in Appendix A.

The values which we use to compute $\Box R_{\mu\nu}$ are as follows:

$$\Gamma^r{}_{tt} = e^{2(\phi-\lambda)}\phi', \qquad \textbf{(3.47)}$$

$$\Gamma^r{}_{\theta\theta} = -re^{-2\lambda}, \qquad \textbf{(3.48)}$$

$$\Gamma^t{}_{tr} = \phi', \qquad \textbf{(3.49)}$$

$$\Gamma^\phi{}_{r\phi} = \frac{1}{r}, \qquad \textbf{(3.50)}$$

$$\Gamma^r{}_{rr} = \lambda', \qquad \textbf{(3.51)}$$

$$\Gamma^r{}_{\phi\phi} = -re^{-2\lambda}\sin^2\theta, \qquad \textbf{(3.52)}$$

$$\Gamma^\theta{}_{r\theta} = \frac{1}{r}, \qquad \textbf{(3.53)}$$

$$\Gamma^\theta{}_{\phi\phi} = -\sin\theta\cos\theta, \qquad \textbf{(3.54)}$$

$$\Gamma^\phi{}_{\theta\phi} = \frac{\cos\theta}{\sin\theta}. \qquad \textbf{(3.55)}$$

Other Levi-Civita connections for spherically symmetric metric vanish.

Metric and Ricci Tensor have only diagonal components. Therefore some indices would be necessarily $t$. Additionally all components of the Ricci tensor depend on $r$. As a result, partial derivative of Ricci tensor with respect to $r$ is non zero and must be calculated. Other partial derivatives are all zero. Thus we get

$$
\begin{aligned}
\Box R_{tt} \;=\;& g_{tt}g^{\rho\lambda}\left(\partial_\lambda\partial_\rho R^t{}_t - \Gamma^\alpha{}_{\lambda\rho}\partial_\alpha R^t{}_t + \Gamma^t{}_{\lambda\beta}\Gamma^\beta{}_{\rho t}R^t{}_t\right) \\
&+g_{tt}g^{\rho\lambda}\left(\Gamma^\alpha{}_{\lambda t}\Gamma^t{}_{\rho\alpha}R^t{}_t - \Gamma^\beta{}_{\lambda t}\Gamma^t{}_{\rho\alpha}R^\alpha{}_\beta - \Gamma^t{}_{\lambda\beta}\Gamma^\alpha{}_{\rho t}R^\beta{}_\alpha\right) \qquad \textbf{(3.56)}
\end{aligned}
$$

Substituting values of $g^{\rho\lambda}$ we obtain

$$\begin{aligned}\Box R_{tt} &= \left(-\Gamma^r{}_{tt}\partial_r R^t{}_t + 2\Gamma^t{}_{tr}\Gamma^r{}_{tt}R^t{}_t - 2\Gamma^r{}_{tt}\Gamma^t{}_{tr}R^r{}_r\right) \\ &+ g_{tt}g^{rr}\left(\partial_r\partial_r R^t{}_t - \Gamma^r{}_{rr}\partial_r R^t{}_t\right) \\ &+ g_{tt}g^{\theta\theta}\left(-\Gamma^r{}_{\theta\theta}\partial_r R^t{}_t\right) + g_{tt}g^{\phi\phi}\left(-\Gamma^r{}_{\phi\phi}\partial_r R^t{}_t\right) \qquad (3.57)\end{aligned}$$

Now plugging in the values of Levi-Civita connections and Ricci tensors we get

$$\begin{aligned}\Box R_{tt} &= e^{2(\phi-\lambda)}\phi'\frac{1}{2}\left(\rho'+3P'\right) - 2\phi' e^{2(\phi-\lambda)}\phi'\left(\rho+P\right) \\ &- e^{2(\phi-\lambda)}\left[-\frac{1}{2}\left(\rho''+3P''\right) + \frac{1}{2}\lambda'\left(\rho'+3P'\right)\right] \\ &+ \frac{1}{2r}e^{2(\phi-\lambda)}\left[\left(\rho'+3P'\right) + \sin^2\theta\left(\rho'+3P'\right)\sin^{-2}\theta\right] \qquad (3.58)\end{aligned}$$

Simplifying this expression we finally find

$$\begin{aligned}\Box R_{tt} &= e^{2(\phi-\lambda)}\left[\frac{1}{2}\left(\rho''+3P''\right) - 2\left(\phi'\right)^2\left(\rho+P\right)\right] \\ &+ e^{2(\phi-\lambda)}\left\{\left[\frac{1}{2}\left(\phi'-\lambda'\right) + \frac{1}{r}\right]\left(\rho'+3P'\right)\right\}. \qquad (3.59)\end{aligned}$$

Using the same method for all components of $\Box R_{\mu\nu}$ , it is possible to show that

$$\begin{aligned}\Box R_{rr} &= \frac{1}{2}\left(\rho''-P''\right) + \left[\frac{1}{2}\left(\phi'-\lambda'\right) + \frac{1}{r}\right]\left(\rho'-P'\right) - 2\left(\phi'\right)^2\left(\rho+P\right), \qquad (3.60) \\ \Box R_{\theta\theta} &= r^2 e^{-2\lambda}\left\{\frac{1}{2}\left(\rho''-P''\right) + \left[\frac{1}{2}\left(\phi'-\lambda'\right) + \frac{1}{r}\right]\left(\rho'-P'\right)\right\}, \qquad (3.61) \\ \Box R_{\phi\phi} &= \sin^2\theta r^2 e^{-2\lambda}\left\{\frac{1}{2}\left(\rho''-P''\right) + \left[\frac{1}{2}\left(\phi'-\lambda'\right) + \frac{1}{r}\right]\left(\rho'-P'\right)\right\}. \qquad (3.62)\end{aligned}$$

## 3.2 TOV Equations

### 3.2.1 The first TOV equation

To determine the first TOV equation we first evaluate *tt* component of EoM (**3.1**), by plugging into it the Equations (**3.4**),(**3.5**),(**3.6**),(**3.7**) and (**3.8**). The *tt* component of EoM (**3.1**) is

$$\begin{aligned}8\pi G T_{tt} &= G_{tt} + \beta\left[-\frac{1}{2}g_{tt}R_{ab}R^{ab} + \nabla^\rho\nabla_\rho R_{tt}\right] \\ &+ \beta\left[-\nabla_t\nabla_t R - 2R_{\sigma tt\alpha}R^{\alpha\sigma} + \frac{1}{2}\Box R g_{tt}\right]. \qquad (3.63)\end{aligned}$$

Using Equations (**3.20**), (**3.23**), (**3.28**), (**3.37**) and (**3.59**) we obtain

$$
\begin{aligned}
8\pi G\rho \;=\; & \frac{1}{r^2}e^{-2\lambda}\left(2r\lambda' - 1 + e^{2\lambda}\right) + \frac{1}{2}\beta\left(\rho^2 + 3P^2\right) + \\
& \beta e^{-2\lambda}\left\{\frac{1}{2}\left(\rho'' + 3P''\right) + \left[\frac{1}{2}\left(\phi' - \lambda'\right) + \frac{1}{r}\right]\left(\rho' + 3P'\right) - 2\left(\phi'\right)^2\left(\rho + P\right)\right\} \\
& + \beta e^{-2\lambda}\phi' R' - 2\beta e^{-2\lambda}\left[\frac{1}{2}\left(\phi'\lambda' - \left(\phi'\right)^2 - \phi''\right) - \frac{1}{r}\phi'\right]\left(\rho - P\right) \\
& - \frac{1}{2}\beta e^{-2\lambda}\left[R'' + \left(\frac{1}{2}\frac{g'}{g} - 2\lambda'\right)R'\right] \qquad \textbf{(3.64)}
\end{aligned}
$$

The above equation contains derivatives $\phi'$, $\lambda'$ etc. as well as hydrodynamic quantities $P$ and $\rho$. The presence of these higher order derivatives precludes expressing the equation in terms of hydrodynamic quantities only. In order to achieve this we use the perturbative approach [15,16] where GR is the zeroth order model of gravity. This method had already been applied to $f(R)$ models of gravity via perturbative constraints at cosmological scales [10,11] and neutron stars with $f(R) = R + \alpha R^{n+1}$ [17,18]. In the perturbative approach, $g_{\mu\nu}$ can be expanded perturbatively in terms of $\beta$:

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + \beta g^{(1)}_{\mu\nu} + O(\beta^2) \qquad \textbf{(3.65)}$$

Accordingly, the metric functions must also be expanded in terms of $\beta$ such as

$$\phi_\beta = \phi + \beta\phi_1 + \cdots \qquad \textbf{(3.66)}$$

and

$$\lambda_\beta = \lambda + \beta\lambda_1 + \cdots . \qquad \textbf{(3.67)}$$

Hydrodynamic quantities on the left hand side of Equation (**3.64**) are then defined perturbatively as:

$$\rho_\beta = \rho + \beta\rho_1 + \cdots \qquad \textbf{(3.68)}$$

and

$$P_\beta = P + \beta P_1 + \cdots . \qquad \textbf{(3.69)}$$

Plugging these expressions into Equation (**3.64**), we obtain

$$
\begin{aligned}
8\pi G\rho_\beta &= \frac{1}{r^2}e^{-2\lambda_\beta}\left(2r\lambda_\beta' - 1 + e^{2\lambda_\beta}\right) + \frac{1}{2}\beta\left(\rho^2 + 3P^2\right) \\
&+\beta e^{-2\lambda}\left[\frac{1}{2}\left(\phi' - \lambda'\right) + \frac{1}{r}\right]\left(\rho' + 3P'\right) \\
&+\beta e^{-2\lambda}\left[\frac{1}{2}\left(\rho'' + 3P''\right) - 2\left(\phi'\right)^2\left(\rho + P\right)\right] \\
&+\beta e^{-2\lambda}\phi' R' - \frac{1}{2}\beta e^{-2\lambda}\left[R'' + \left(\frac{1}{2}\frac{g'}{g} - 2\lambda'\right)R'\right] \\
&-2\beta e^{-2\lambda}\left[\frac{1}{2}\left(\phi'\lambda' - \left(\phi'\right)^2 - \phi''\right) - \frac{1}{r}\phi'\right]\left(\rho - P\right) \qquad \textbf{(3.70)}
\end{aligned}
$$

It can easily be seen that

$$
1 - e^{-2\lambda_\beta} + 2r\lambda_\beta' e^{-2\lambda_\beta} = 1 - \frac{d}{dr}\left(re^{-2\lambda_\beta}\right). \qquad \textbf{(3.71)}
$$

We define a mass parameter $M_\beta$ by the relation $e^{-2\lambda_\beta} = 1 - M_\beta/r$ and the equation above becomes

$$
1 - e^{-2\lambda_\beta} + 2r\lambda_\beta' e^{-2\lambda_\beta} = 1 - \frac{d}{dr}\left(r - M_\beta\right). \qquad \textbf{(3.72)}
$$

It is possible to arrange the above equation as

$$
\frac{dM_\beta}{dr} = e^{-2\lambda_\beta}\left[2r\lambda_\beta' - 1 + e^{2\lambda_\beta}\right]. \qquad \textbf{(3.73)}
$$

By plugging Equation (**3.73**) into Equation (**3.70**) we obtain

$$
\begin{aligned}
8\pi G\rho_\beta &= \frac{1}{r^2}\frac{dM_\beta}{dr} + \frac{1}{2}\beta\left(\rho^2 + 3P^2\right) + \beta e^{-2\lambda}\phi' R' \\
&+\beta e^{-2\lambda}\left[\frac{1}{2}\left(\phi' - \lambda'\right) + \frac{1}{r}\right]\left(\rho' + 3P'\right) \\
&-2\beta e^{-2\lambda}\left(\phi'\right)^2\left(\rho + P\right) + \frac{1}{2}\beta e^{-2\lambda}\left(\rho'' + 3P''\right) \\
&-\frac{1}{2}\beta e^{-2\lambda}\left[R'' + \left(\frac{1}{2}\frac{g'}{g} - 2\lambda'\right)R'\right] \\
&-2\beta e^{-2\lambda}\left[\frac{1}{2}\left(\phi'\lambda' - \left(\phi'\right)^2 - \phi''\right) - \frac{1}{r}\phi'\right]\left(\rho - P\right). \qquad \textbf{(3.74)}
\end{aligned}
$$

In the perturbative approach the derivatives like $\phi'$, $\lambda'$ etc. can be calculated from the zeroth order gravity model, general relativity. Thus, in Equation (**3.74**) all terms which are multiplied with $\beta$ can be rearranged in terms of general relativistic expressions by ignoring second order terms. In Appendix A we present all components of the Ricci tensor and Einstein tensor for spherically symmetric metric. Recalling these

$$\begin{aligned}
R_{tt} &= e^{2(\phi-\lambda)}\left[\phi'' + \left(\phi'\right)^2 - \phi'\lambda' + \frac{2}{r}\phi'\right] \\
R_{rr} &= -\left[\phi'' + \left(\phi'\right)^2 - \phi'\lambda' - \frac{2}{r}\lambda'\right] \\
R_{\theta\theta} &= e^{-2\lambda}\left[r\left(\lambda' - \phi'\right) - 1\right] + 1 \\
R_{\phi\phi} &= \sin^2\theta e^{-2\lambda}\left[r\left(\lambda' - \phi'\right) - 1\right] + \sin^2\theta \\
R_{\phi\phi} &= \sin^2\theta R_{\theta\theta}.
\end{aligned} \tag{3.75}$$

We plug all components of the Ricci tensor into the definition of Ricci scalar,

$$R = g^{tt}R_{tt} + g^{rr}R_{rr} + g^{\theta\theta}R_{\theta\theta} + g^{\phi\phi}R_{\phi\phi}. \tag{3.76}$$

and obtain

$$R = 2e^{-2\lambda}\left[-\phi'' - \left(\phi'\right)^2 + \phi'\lambda' + \frac{2}{r}\left(\lambda' - \phi'\right) - \frac{1}{r^2}\right] + \frac{2}{r^2} \tag{3.77}$$

which can be arranged as

$$\frac{R}{2} - \frac{1}{r^2} + \frac{1}{r^2}e^{-2\lambda} - \frac{2}{r}\lambda' e^{-2\lambda} = e^{-2\lambda}\left[\phi'\lambda' - \phi'' - \left(\phi'\right)^2 - \frac{2}{r}\phi'\right]. \tag{3.78}$$

The Ricci scalar in Equation (**3.78**) can be calculated from the trace of the Einstein's field equation as

$$R = \rho - 3P. \tag{3.79}$$

Now, from the definition

$$e^{-2\lambda} = 1 - \frac{M}{r} \tag{3.80}$$

we obtain

$$\lambda' e^{-2\lambda} = \frac{1}{2}\left(\rho r - \frac{M}{r^2}\right) \tag{3.81}$$

by taking derivative. Plugging Equations (**3.79**) and (**3.81**) into Equation (**3.78**) we get

$$e^{-2\lambda}\left[\phi'\lambda' - \phi'' - \left(\phi'\right)^2 - \frac{2}{r}\phi'\right] = -\frac{1}{2}\left(\rho + 3P\right) \tag{3.82}$$

Now we need to express the derivatives of $\phi(r)$ in terms of hydrodynamic quantities. For that firstly we use

$$\begin{aligned}
G^t{}_t &= -e^{-2\lambda}\frac{1}{r^2}\left(2r\lambda' - 1 + e^{2\lambda}\right) \\
&= -\rho \\
G^r{}_r &= e^{-2\lambda}\frac{1}{r^2}\left(2r\phi' + 1 - e^{2\lambda}\right) \\
&= P
\end{aligned} \tag{3.83}$$

By subtracting these equations side by side we get

$$G^r{}_r - G^t{}_t = \frac{2}{r}e^{-2\lambda}\left(\phi' + \lambda'\right) = \rho + P. \tag{3.84}$$

Using Equation (**3.81**) we find expression for $\phi'$ as

$$e^{-2\lambda}\phi' = \frac{1}{2}Pr + \frac{1}{2}\frac{M}{r^2}. \tag{3.85}$$

In order to compute $e^{-2\lambda}\left[\frac{1}{2}\left(\phi' - \lambda'\right) + \frac{1}{r}\right]$ in Equation (**3.74**) we can use Equations (**3.85**), (**3.81**) and (**3.80**) to obtain

$$e^{-2\lambda}\left[\frac{1}{2}\left(\phi' - \lambda'\right) + \frac{1}{r}\right] = \frac{1}{4}r\left(P - \rho\right) - \frac{1}{2}\frac{M}{r^2} + \frac{1}{r} \tag{3.86}$$

In order to compute $e^{-2\lambda}\left(\phi'\right)^2$ in Equation (**3.74**) we refer to Equations (**3.80**) and (**3.85**) to obtain

$$e^{-2\lambda}\left(\phi'\right)^2 = \frac{1}{4}\frac{1}{(r-M)}\left(P^2r^3 + \frac{M^2}{r^3} + 2PM\right). \tag{3.87}$$

Another term we would like to compute is $e^{-2\lambda}\left[R'' + \left(\frac{1}{2}\frac{g'}{g} - 2\lambda'\right)R'\right]$ in Equation (**3.74**). Since $R = \rho - 3P$, we infer that $R' = \rho' - 3P'$ and $R'' = \rho'' - 3P''$. Determinant of the metric tensor and its derivative with respect to $r$ are

$$g = \det g_{\mu\nu} = \begin{vmatrix} -e^{2\phi} & 0 & 0 & 0 \\ 0 & e^{2\lambda} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{vmatrix} = -r^4\sin^2\theta e^{2(\phi+\lambda)}, \tag{3.88}$$

$$\begin{aligned} g' &= -4r^3\sin^2\theta e^{2(\phi+\lambda)} - 2r^4\sin^2\theta\left(\phi' + \lambda'\right)e^{2(\phi+\lambda)} \\ &= -r^4\sin^2\theta e^{2(\phi+\lambda)}\left[\frac{4}{r} + 2\left(\phi' + \lambda'\right)\right], \end{aligned} \tag{3.89}$$

therefore

$$\frac{g'}{g} = 2\left(\frac{2}{r} + \phi' + \lambda'\right). \tag{3.90}$$

Bringing all these together we find that

$$\begin{aligned} e^{-2\lambda}\left[R'' + \left(\frac{1}{2}\frac{g'}{g} - 2\lambda'\right)R'\right] &= \left(1 - \frac{M}{r}\right)\left(\rho'' - 3P''\right) \\ &\quad + \left[\frac{2}{r} - \frac{M}{r^2} + \frac{1}{2}r\left(P - \rho\right)\right]\left(\rho' - 3P'\right). \end{aligned} \tag{3.91}$$

We now plug in the above equation, as well as all other necessary terms calculated on the way, into Equation (**3.74**) to obtain

$$\begin{aligned}
8\pi G\rho_\beta &= \frac{1}{r^2}\frac{dM_\beta}{dr}+\frac{1}{2}\beta\left(\rho^2+3P^2\right)+\frac{\beta}{2}\left(1-\frac{M}{r}\right)\left(\rho''+3P''\right)\\
&+\beta\left(\frac{1}{4}Pr-\frac{1}{4}\rho r-\frac{1}{2}\frac{M}{r^2}+\frac{1}{r}\right)\left(\rho'+3P'\right)\\
&-\beta\frac{1}{2}\frac{1}{(r-M)}\left(P^2r^3+\frac{M^2}{r^3}+2PM\right)(\rho+P)\\
&+\beta\left(\frac{1}{2}Pr+\frac{1}{2}\frac{M}{r^2}\right)\left(\rho'-3P'\right)+\frac{\beta}{2}(\rho+3P)(\rho-P)\\
&-\frac{\beta}{2}\left(1-\frac{M}{r}\right)\left(\rho''-3P''\right)\\
&-\frac{\beta}{2}\left(\frac{2}{r}-\frac{M}{r^2}+\frac{1}{2}Pr-\frac{1}{2}\rho r\right)\left(\rho'-3P'\right). \qquad (3.92)
\end{aligned}$$

By setting $G=1$ and $c=1$ and arranging we obtain

$$\begin{aligned}
8\pi\rho_\beta &= \frac{1}{r^2}\frac{dM_\beta}{dr}+\beta\left(\rho^2+P\rho\right)+\beta\left(1-\frac{M}{r}\right)3P''\\
&-\frac{\beta}{2}\frac{1}{(r-M)}\left(P^2r^3+\frac{M^2}{r^3}+2PM\right)(\rho+P)\\
&+\beta\left(\frac{1}{2}Pr+\frac{1}{2}\frac{M}{r^2}\right)\rho'+\beta\left(-\frac{1}{2}\rho r-\frac{3}{2}\frac{M}{r^2}+\frac{2}{r}\right)3P'. \qquad (3.93)
\end{aligned}$$

Finally, we arrange this equation as

$$\begin{aligned}
\frac{dM_\beta}{dr} &= 8\pi\rho_\beta r^2-\beta r^2\left(\rho^2+P\rho\right)-\beta r^2\left(1-\frac{M}{r}\right)3P''\\
&+\beta\frac{r^2}{2(r-M)}\left(P^2r^3+\frac{M^2}{r^3}+2PM\right)(\rho+P)\\
&-\beta\frac{r^2}{2}\left(Pr+\frac{M}{r^2}\right)\rho'+\beta\frac{3}{2}r^2\left(\rho r+3\frac{M}{r^2}-\frac{4}{r}\right)P' \qquad (3.94)
\end{aligned}$$

which is the first modified TOV equation.

### 3.2.2 The second TOV equation

We start with $rr$ component of the EoM (**3.1**) calculated in the previous Chapter:

$$\begin{aligned}
8\pi GT_{rr} &= G_{rr}+\beta\left(-\frac{1}{2}g_{rr}R_{ab}R^{ab}+\nabla^\rho\nabla_\rho R_{rr}\right)\\
&+\beta\left(-\nabla_r\nabla_r R-2R_{\sigma rr\alpha}R^{\alpha\sigma}+\frac{1}{2}\Box R g_{rr}\right) \qquad (3.95)
\end{aligned}$$

Specific component of Einstein's tensor in this background is

$$G_{rr}=\frac{1}{r^2}\left(2r\phi'+1-e^{2\lambda}\right). \qquad (3.96)$$

Using calculated forms (**3.20**), (**3.23**), (**3.32**), (**3.39**) and (**3.60**) of terms of (**3.95**) we find

$$\begin{aligned} 8\pi GP &= \frac{1}{r^2}\left(2r\phi'+1-e^{2\lambda}\right)e^{-2\lambda}-\frac{1}{2}\beta\left(\rho^2+3P^2\right)+\frac{1}{2}\beta e^{-2\lambda}\left(\rho''-P''\right) \\ &+\beta e^{-2\lambda}\left[\frac{1}{2}\left(\phi'-\lambda'\right)+\frac{1}{r}\right]\left(\rho'-P'\right)-2\beta e^{-2\lambda}\left(\phi'\right)^2\left(\rho+P\right) \\ &-\beta e^{-2\lambda}\left(R''-\lambda' R'\right)-\beta e^{-2\lambda}\left(\rho+3P\right)\left[\phi'\lambda'-\left(\phi'\right)^2-\phi''\right] \\ &+2\beta e^{-2\lambda}\frac{1}{r}\lambda'\left(\rho-P\right)+\frac{\beta}{2}e^{-2\lambda}\left[R''+\left(\frac{1}{2}\frac{g'}{g}-2\lambda'\right)R'\right]. \qquad (\mathbf{3.97}) \end{aligned}$$

Plugging in the hydrodynamic equivalent of all terms like $g'/g$ etc. which we have calculated previously we obtain

$$\begin{aligned} 8\pi GP &= \frac{1}{r^2}\left(2r\phi'+1-e^{2\lambda}\right)e^{-2\lambda}-\frac{1}{2}\beta\left(\rho^2+3P^2\right)+\frac{1}{2}\beta e^{-2\lambda}\left(\rho''-P''\right) \\ &+\beta e^{-2\lambda}\left[\frac{1}{2}\left(\phi'-\lambda'\right)+\frac{1}{r}\right]\left(\rho'-P'\right)-2\beta e^{-2\lambda}\left(\phi'\right)^2\left(\rho+P\right) \\ &-\beta e^{-2\lambda}\left(\rho+3P\right)\left[\phi'\lambda'-\left(\phi'\right)^2-\phi''\right]+2\beta e^{-2\lambda}\frac{1}{r}\lambda'\left(\rho-P\right) \\ &-\frac{1}{2}\beta e^{-2\lambda}\left(\rho''-3P''\right)+\frac{1}{2}\beta e^{-2\lambda}\left(\frac{2}{r}+\phi'+\lambda'\right)\left(\rho'-3P'\right) \qquad (\mathbf{3.98}) \end{aligned}$$

We again deal with this equation perturbatively. We assume all functions multiplied with $\beta$ have general relativistic values and all the other functions have power expansion in $\beta$. Setting $G=1$ we get

$$\begin{aligned} 8\pi P_\beta &= \frac{1}{r^2}\left(2r\phi'_\beta+1-e^{2\lambda_\beta}\right)e^{-2\lambda_\beta}-\frac{1}{2}\beta\left(\rho^2+3P^2\right)+\frac{1}{2}\beta e^{-2\lambda}\left(\rho''-P''\right) \\ &+\beta e^{-2\lambda}\left[\frac{1}{2}\left(\phi'-\lambda'\right)+\frac{1}{r}\right]\left(\rho'-P'\right)-2\beta e^{-2\lambda}\left(\phi'\right)^2\left(\rho+P\right) \\ &-\beta e^{-2\lambda}\left(\rho+3P\right)\left[\phi'\lambda'-\left(\phi'\right)^2-\phi''\right]+2\beta e^{-2\lambda}\frac{1}{r}\lambda'\left(\rho-P\right) \\ &-\frac{1}{2}\beta e^{-2\lambda}\left(\rho''-3P''\right)+\frac{1}{2}\beta e^{-2\lambda}\left(\frac{2}{r}+\phi'+\lambda'\right)\left(\rho'-3P'\right). \qquad (\mathbf{3.99}) \end{aligned}$$

We want to calculate $e^{-2\lambda}\left[\phi'-\lambda'+\frac{2}{r}\right]$ in the above equation. Using (**3.81**) and (**3.85**) we obtain

$$e^{-2\lambda}\left[\phi'-\lambda'+\frac{2}{r}\right]=\frac{r}{2}\left(P-\rho\right)-\frac{M}{r^2}+\frac{2}{r}. \qquad (\mathbf{3.100})$$

We also calculate $e^{-2\lambda}\left[\phi'\lambda'-\left(\phi'\right)^2-\phi''\right]$ by using the expressions of its terms in terms of hydrodynamic quantities (**3.82**), (**3.85**) as

$$e^{-2\lambda}\left[\phi'\lambda'-\left(\phi'\right)^2-\phi''\right]=-\frac{1}{2}\rho-\frac{1}{2}P+\frac{M}{r^3}. \qquad (\mathbf{3.101})$$

Similarly,

$$e^{-2\lambda}\left(\phi'+\lambda'+\frac{2}{r}\right)=\frac{1}{2}Pr+\frac{1}{2}\rho r+\frac{2}{r}-2\frac{M}{r^2} \qquad \textbf{(3.102)}$$

Plugging in the last three equations and Equation (**3.87**) into Equation (**3.99**) we obtain

$$\begin{aligned}
8\pi P_\beta \;=\;& \frac{1}{r^2}\left(2r\phi'_\beta+1-e^{2\lambda_\beta}\right)e^{-2\lambda_\beta}-\frac{1}{2}\beta\left(\rho^2+3P^2\right)\\
&+\frac{1}{2}\beta\left(1-\frac{M}{r}\right)\left(\rho''-P''\right)+2\beta\frac{1}{r}\left(\frac{1}{2}\rho r-\frac{1}{2}\frac{M}{r^2}\right)(\rho-P)\\
&+\frac{\beta}{2}\left(\frac{1}{2}Pr-\frac{1}{2}\rho r-\frac{M}{r^2}+\frac{2}{r}\right)\left(\rho'-P'\right)\\
&-\frac{\beta}{2}\frac{1}{(r-M)}\left(P^2r^3+\frac{M^2}{r^3}+2PM\right)(\rho+P)\\
&-\beta\left(-\frac{1}{2}\rho-\frac{1}{2}P+\frac{M}{r^3}\right)(\rho+3P)\\
&-\frac{1}{2}\beta\left(1-\frac{M}{r}\right)\left(\rho''-3P''\right)\\
&+\frac{1}{2}\beta\left(\frac{1}{2}Pr+\frac{1}{2}\rho r+\frac{2}{r}-2\frac{M}{r^2}\right)\left(\rho'-3P'\right) \qquad \textbf{(3.103)}
\end{aligned}$$

From this equation it follows that

$$\begin{aligned}
\phi'_\beta \;=\;& \frac{1}{2\left(r-M_\beta\right)}\left[\frac{M_\beta}{r}+8\pi P_\beta-\beta r^2\left(1-\frac{M}{r}\right)P''\right]\\
&-\beta r^2\frac{1}{2\left(r-M_\beta\right)}\left[2\rho^2+3P^2+P\rho-2\frac{M}{r^3}(\rho+P)\right]\\
&-\beta r^2\frac{1}{2\left(r-M_\beta\right)}\left(\frac{1}{2}Pr+\frac{2}{r}-\frac{3}{2}\frac{M}{r^2}\right)\rho'\\
&-\beta r^2\frac{1}{2\left(r-M_\beta\right)}\left(Pr+\frac{1}{2}\rho r-\frac{7}{2}\frac{M}{r^2}+\frac{4}{r}\right)P'\\
&+\frac{1}{2}\beta\frac{1}{2\left(r-M_\beta\right)}\frac{r^2}{(r-M)}\left(P^2r^3+\frac{M^2}{r^3}+2PM\right)(\rho+P). \qquad \textbf{(3.104)}
\end{aligned}$$

Hydrostatic equilibrium equation is the conservation equation of energy-momentum tensor, $\nabla_\mu T^\mu{}_\nu=0$ which has the same form regardless of the gravity theory:

$$\frac{dP}{dr}=-(\rho+P)\phi'_\beta. \qquad \textbf{(3.105)}$$

This is the second modified TOV equation together with equation (**3.104**). This equation and Equation (**3.94**) will be solved in the next Chapter by supplementing them with a relation between $P$ and $\rho$.

# 4. SOLUTION OF THE NEUTRON STAR STRUCTURE

In this Chapter we solve the modified TOV equations with realistic equations of state (EoS) appropriate for neutron stars. We will use cgs units in solving the equations. The equations we obtained were written in natural units where $c = G = 1$. Before solving them numerically we will convert the equations to cgs units by plugging the physical constants

$$c = 2.99792458 \times 10^{10}\,\mathrm{cm\,s^{-1}} \tag{4.1}$$

and

$$G = 6.67259 \times 10^{-11}\,\mathrm{cm^3\,g^{-1}s^{-2}}. \tag{4.2}$$

in appropriate places.

The dimension of the gravitational constant is

$$[G] = L^3 M^{-1} T^{-2} \tag{4.3}$$

The dimension of the pressure is

$$[P] = M L^{-1} T^{-2} \tag{4.4}$$

The dimension of the coupling constant $\beta$ is

$$[\beta] = L^2 \tag{4.5}$$

Accordingly, the mass conservation becomes

$$\frac{dm}{dr} = 4\pi r^2 \rho + \frac{1}{2}\beta r^2 K \tag{4.6}$$

where

$$
\begin{aligned}
K &= -\left(1+\frac{P}{\rho c^2}\right)\frac{G\rho^2}{c^2} \\
&\quad -\left(1-\frac{2Gm}{c^2 r}\right)3\frac{P''}{c^2} \\
&\quad +\frac{Gm}{rc^2}\frac{GP\rho}{2c^4}\left(1+\frac{P}{\rho c^2}\right)\left(\frac{Pr^3}{mc^2}+\frac{4mc^2}{r^3 P}+4\right)\left(1-\frac{2Gm}{rc^2}\right)^{-1} \\
&\quad -\left(1+\frac{2mc^2}{r^3 P}\right)\frac{GrP\rho'}{2c^4} \\
&\quad +\left(\frac{\rho c^2}{P}+\frac{6mc^2}{r^3 P}-\frac{4c^4}{Gr^2 P}\right)\frac{3Gr^2 PP'}{2rc^6}
\end{aligned} \tag{4.7}
$$

The hydrostatic equilibrium equation becomes

$$
\frac{dP}{dr} = -\frac{Gm\rho\left(1+\frac{P}{\rho c^2}\right)}{r^2\left(1-\frac{2Gm}{c^2 r}\right)}\left(1+\frac{4\pi r^3 P}{m_\alpha c^2}+\frac{1}{2}\beta r^2 H\right) \tag{4.8}
$$

where

$$
\begin{aligned}
H = - \quad & (1-\frac{2Gm}{rc^2})\frac{rP''}{m_\alpha c^2} \\
& -\left(\frac{\rho^2 c^4}{P^2}+\frac{3}{2}+\frac{1}{2}\frac{\rho c^2}{P}-\frac{2mc^2}{r^3 P}\frac{\rho c^2}{P}-\frac{2mc^2}{r^3 P}\right)\frac{2GrP^2}{m_\alpha c^6} \\
& -(\frac{1}{2}+\frac{2c^4}{Gr^2 P}-\frac{3mc^2}{r^3 P})\frac{Gr^2 P\rho'}{m_\alpha c^4} \\
& +\left(1+\frac{1}{2}\frac{\rho c^2}{P}-\frac{7Mc^2}{r^3 P}+\frac{4c^4}{Gr^2 P}\right)\frac{GP' r^2 P}{m_\alpha c^6} \\
& +\frac{\frac{1}{4}\frac{Pr^3}{mc^2}+\frac{mc^2}{r^3 P}+1}{\left(1-\frac{2Gm}{rc^2}\right)}\left(1+\frac{P}{\rho c^2}\right)\frac{P\rho}{c^2}\frac{2mG^2}{m_\alpha c^4}
\end{aligned} \tag{4.9}
$$

Some of the commonly appearing *dimensionless* terms in these equations are

$$
u_1 \equiv \frac{Gm}{c^2 r}, \qquad u_2 \equiv \frac{P}{\rho c^2}, \qquad u_3 \equiv \frac{r^3 P}{mc^2}, \qquad u_4 \equiv \frac{c^4}{Gr^2 P} \tag{4.10}
$$

In order to eliminate errors in coding these lengthy equations, we use these definitions to shorten the equations. This also speeds up the code by reducing the number of calculations. We thus obtain

$$
\begin{aligned}
\frac{Gr^4}{c^2}K &= -\frac{1+u_2}{u_2^2 u_4^2}-\frac{3(1-2u_1)}{u_4}\left(\frac{P'' r^2}{P}\right) \\
&\quad +\frac{\frac{1}{2}u_1(1+u_2)\left(u_3+\frac{4}{u_3}+4\right)}{u_2 u_4^2(1-2u_1)} \\
&\quad -\left(\frac{r\rho'}{\rho}\right)\frac{\frac{1}{2}+\frac{1}{u_3}}{u_2 u_4^2}+\left(\frac{P' r}{P}\right)\frac{\frac{3}{2}\left(\frac{1}{u_2 u_4}+\frac{6}{u_3 u_4}-4\right)}{u_4}
\end{aligned} \tag{4.11}
$$

and

$$
\begin{aligned}
Hr^4 = - \quad & (1-2u_1)u_3\frac{r^2P''}{P} \\
& -\left(\frac{1}{u_2^2}+\frac{3}{2}+\frac{1}{2u_2}-\frac{2}{u_3u_2}-\frac{2}{u_3}\right)\frac{2u_3}{u_4} \\
& -(\frac{1}{2}+2u_4-\frac{3}{u_3})\frac{u_3}{u_2u_4}\frac{r\rho'}{\rho} \\
& +\left(1+\frac{1}{2u_2}-\frac{7}{u_3}+4u_4\right)u_3^2u_1\frac{rP'}{P} \\
& +\left(\frac{1+u_2}{1-2u_1}\right)\frac{2}{u_2u_4^2} \qquad \textbf{(4.12)}
\end{aligned}
$$

Note also that $rP'/P$, $r^2P''/P$ and $r\rho'/\rho$, appearing in these equations, are also dimensionless combinations.

## 4.1 Higher Derivatives

Higher derivatives like $P'$, $\rho'$ and $P''$ are calculated by using the TOV equations obtained in GR. As such terms come only in the perturbative term, the error in employing GR instead of the full theory is of the order of $\beta^2$.

We start with the usual TOV equation in GR

$$
\begin{aligned}
P' &= -\frac{Gm\rho}{r^2}\left(1+\frac{P}{\rho c^2}\right)\left(1+\frac{4\pi r^3P}{mc^2}\right)\left(1-\frac{2Gm}{c^2r}\right)^{-1} \\
&= -\frac{\rho c^2}{r}u_1\left(1+u_2\right)\left(1+4\pi u_3\right)\left(1-2u_1\right)^{-1} \qquad \textbf{(4.13)}
\end{aligned}
$$

Taking the derivative of this with respect to $r$ we get

$$
\begin{aligned}
P'' = & -G\left(\frac{(m'\rho+m\rho')r^2-m\rho 2r}{r^4}\right)\left(1+\frac{P}{\rho c^2}\right)\left(1+\frac{4\pi r^3P}{mc^2}\right)\left(1-\frac{2Gm}{c^2r}\right)^{-1} \\
& -\frac{Gm\rho}{r^2}\left(\frac{P'\rho-P\rho'}{\rho^2c^2}\right)\left(1+\frac{4\pi r^3P}{mc^2}\right)\left(1-\frac{2Gm}{c^2r}\right)^{-1} \\
& -\frac{Gm\rho}{r^2}\left(1+\frac{P}{\rho c^2}\right)\left(\frac{4\pi}{c^2}\frac{\left(3r^2P+r^3P'\right)m-r^3Pm'}{m^2}\right)\left(1-\frac{2Gm}{c^2r}\right)^{-1} \\
& -\frac{Gm\rho}{r^2}\left(1+\frac{P}{\rho c^2}\right)\left(1+\frac{4\pi r^3P}{mc^2}\right)\left[\left(1-\frac{2Gm}{c^2r}\right)^{-2}\frac{2G}{c^2}\left(\frac{m'r-m}{r^2}\right)\right] \qquad \textbf{(4.14)}
\end{aligned}
$$

This can be written more simply as

$$
\begin{aligned}
P'' = & P'\left[\left(\frac{m'}{m}+\frac{\rho'}{\rho}-\frac{2}{r}\right)+\frac{u_2}{1+u_2}\left(\frac{P'}{P}-\frac{\rho'}{\rho}\right)\right] \\
& +P'\left[\frac{4\pi u_3}{1+4\pi u_3}\left(\frac{3}{r}+\frac{P'}{P}-\frac{m'}{m}\right)+\frac{2u_1}{1-2u_1}\left(\frac{m'}{m}-\frac{1}{r}\right)\right] \qquad \textbf{(4.15)}
\end{aligned}
$$

where

$$m' = 4\pi r^2 \rho \tag{4.16}$$

and

$$\rho' = \frac{d\rho}{dP}\frac{dP}{dr} = \frac{P'}{dP/d\rho} \tag{4.17}$$

Here $dP/d\rho$ is to be calculated through the equation of state, $P = P(\rho)$. As we will see in the next section we have $\log P(\log \rho)$ rather than $P(\rho)$. It is then easier to evaluate

$$\gamma = \frac{d\log P}{d\log \rho} \tag{4.18}$$

As this equals

$$\frac{d\log P}{d\log \rho} = \frac{\rho}{P}\frac{dP}{d\rho} \tag{4.19}$$

we have

$$\rho' = \frac{\rho P'}{\gamma P} \tag{4.20}$$

where $P'$ is calculated from hydrostatic equilibrium equation in GR.

### 4.2 Equation of State

In order to solve the hydrostatic equilibrium equations (**4.6**) and (**4.8**) we must supplement them with an equation of state (EoS), a relation between density and pressure

$$P = P(\rho). \tag{4.21}$$

Neutron stars are the most dense forms of matter and the EoS of matter at such high densities is not well constrained. Different EoS lead to different mass-radius (M-R) relations. As a result we have to solve the neutron star structure for a number of EoS. In this thesis we present results for 6 representative EoS. We use an analytical representation of $\log P(\log \rho)$ for all the EoS obtained by fitting the tabulated data following the method described in [19]. These are FPS [20] AP4 [21], SLy [22], MS1 [23], MPA1 [24] and GS1 [25]. These EoS are described in [26].

### 4.3 Numerical Method

We numerically integrate Equations (**4.6**) and (**4.8**) supplemented by EoS employing a Runge-Kutta scheme with fixed step size of $\Delta r = 0.1$ km starting from the center of the star for a certain value of central density, $\rho_c$. We identify the surface of the star as the point where pressure drops to a very small value and record the mass contained inside this radius as the mass of the star.

We change the central density $\rho_c$ from $2 \times 10^{14}$ g cm$^{-3}$ to $1 \times 10^{16}$ g cm$^{-3}$ in 200 logarithmically equal steps to obtain a sequence of equilibrium configurations. We record the mass and radius for each central density. This allows us to obtain a mass-radius (M-R) relation for a certain EoS. We then repeat this procedure for a range of $\beta$ to see the effect of the perturbative term we added to the Lagrangian.

### 4.4 Observational Constraints on the Mass-Radius Relation

Recently, the authors of [27] showed that the measurement of masses and radii of three neutron stars are sufficient for constraining the pressure of nuclear matter at densities a few times the density of nuclear saturation. These data are provided by the measurements on the neutron stars EXO 1745-248 [28], 4U 1608-52 [29] and 4U 1820-30 [30] by the methods described in the cited papers. We use the constraints on the M-R relation of neutron stars given in [1], which is a union of these three constraints.[1] The constraint of [1] is shown in all M-R plots as the region bounded by the thin black line.

Apart from the above constraint, the recent measurement [2] of the mass of the neutron star PSR J1614-2230 with $1.97 \pm 0.04 M_{\odot}$ provides a stringent constraint on any M-R relation that can be obtained with a combination of $\beta$ and EoS. This constraint is shown as the horizontal black line with its error shown in grey. Any viable combination of $\beta$ and EoS must yield a M-R relation with a maximum mass exceeding this measured mass.

The constraints on the M-R relation obtained by [1] and the 2 solar mass neutron star PSR J1614-2230 exclude many of the possible EoS' if one assumes GR as the ultimate classical theory of gravity. In the gravity model employed here, the value of $\beta$ provides a new degree of freedom such that some of the EoS', which are excluded within the framework of GR, can now be reconciled with the observations for certain values of $\beta$. In the following we discuss this for all EoS' individually. To save space in the figures, we define the parameter $\beta_{11} \equiv \beta / 10^{11}$ cm$^2$ . We show the stable configurations

[1]For a critic of these constraints see [31].

($dM/d\rho_c > 0$) with solid lines and the unstable configurations with dashed lines of the same color.

### 4.5 Results: The effect of $\beta$ on the M-R relation

We have determined the M-R relation for each EoS for a range of $\beta$ values. In figures, we show our results for these 6 representative EoS'. Results for each EoS are summarized below.

#### 4.5.1 FPS

The central density versus the mass of the neutron star relation $\rho_c - M$ and M-R relation are shown in Figures 4.1 and 4.2, respectively. For FPS, the maximum mass that a neutron star can have, within GR, is about $1.8\,M_\odot$ and is less than the measured mass of PSR J1614-2230. This means FPS can not represent the EoS of neutron stars in GR ($\beta = 0$).

The maximum mass increases with decreasing value of $\beta$ and we find that, for $\beta_{11} < -2$, the maximum mass becomes $M_{\max} \simeq 2M_\odot$. We thus find that FPS is consistent with the measurement of the maximum mass for $\beta < -2 \times 10^{11}$ cm$^2$.

Variations in the M-R relation comparable to employing different EoS' can be obtained for $|\beta| \sim 10^{11}$ cm$^2$. Using $\beta \lesssim 10^{11}$ cm$^2$ gives M-R relations that can not be distinguished from the GR result on this plot.

Interestingly, we find that for $\beta = -2 \times 10^{11}$ cm$^2$ there exists a new branch of stable solution which does not have a counterpart in general relativity.

#### 4.5.2 SLy

The central density versus the mass of the neutron star relation, $\rho_c - M$, and M-R relation are shown in Figures 4.3 and 4.4, respectively. SLy is consistent with both constraints within the framework of GR.

For $\beta > 2 \times 10^{11}$ cm$^2$, however, we see that $M_{\max}$ is less than the measured mass of PSR J1614-2230. We conclude for the gravity model employed here that SLy is consistent with the observations only if $\beta < 2 \times 10^{11}$ cm$^2$. Again, we see for $\beta = -2 \times 10^{11}$ cm$^2$ that a new stable solution branch exists for highest densities.

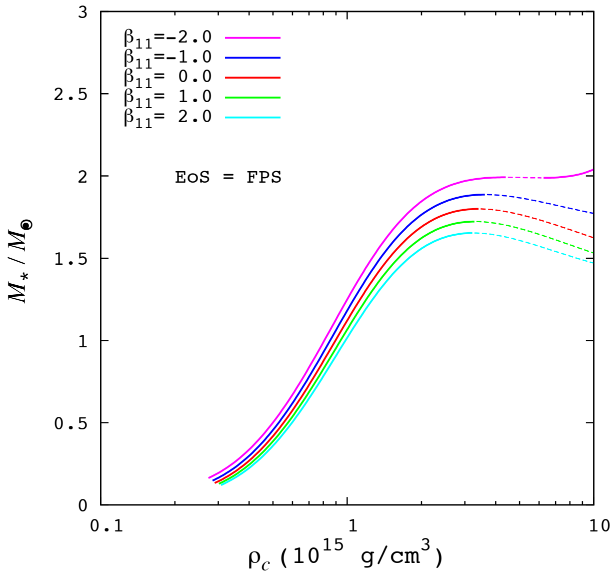


**Figure 4.1**: The $\rho_c - M$ relation for FPS. Each solid line corresponds to a stable configuration for a specific value of $\beta$. Dashed lines show the solutions for unstable configurations ($dM/d\rho_c < 0$). The red line ($\beta = 0$) shows the result for GR.

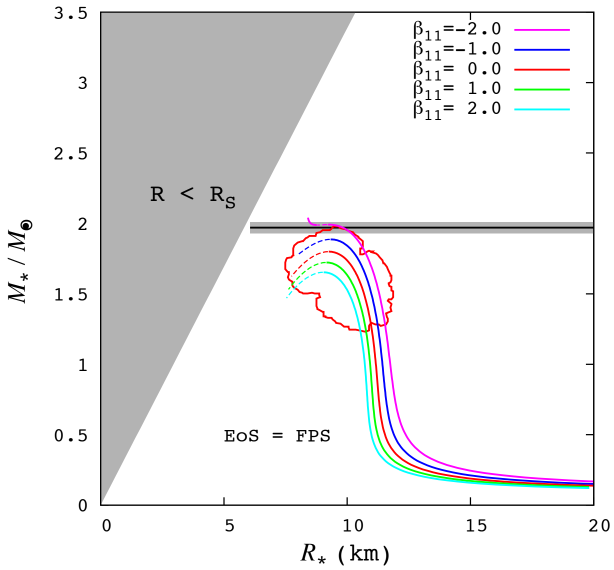


**Figure 4.2**: M-R relation for FPS. The observational constraints of [1] is shown with the thin red contour; the measured mass $M = 1.97 \pm 0.04 M_\odot$ of PSR J1614-2230 [2] is shown as the horizontal black line with grey errorbar. Each solid line corresponds to a stable configuration for a specific value of $\beta$. Dashed lines show the solutions for unstable configurations ($dM/d\rho_c < 0$). The grey shaded region shows where the total mass would be enclosed within its Schwarzschild radius. The red line ($\beta = 0$) shows the result for GR. $M_{\max}$ and $R_{\min}$ increase for decreasing values of $\beta$.

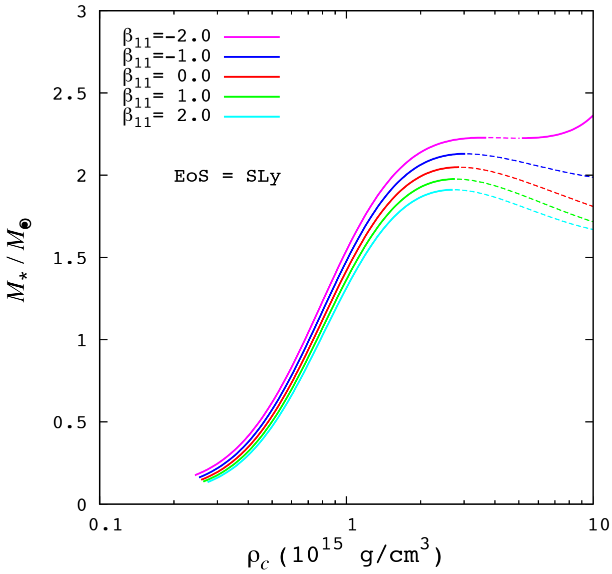


**Figure 4.3**: The $\rho_c - M$ relation for the SLy. The notation in the figure is the same as that of Figure 4.1 and the results are discussed in the text.

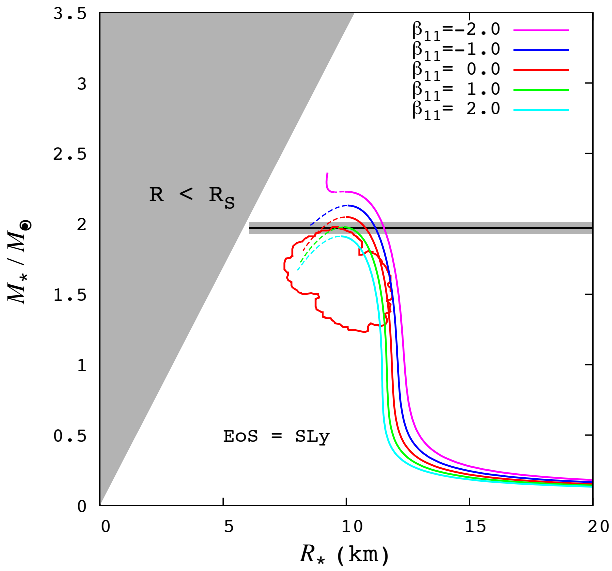


**Figure 4.4**: M-R relation for the SLy. The notation in the figure is the same as that of Figure 4.2 and the results are discussed in the text.

### 4.5.3 AP4

The central density versus the mass of the neutron star relation, $\rho_c - M$, and M-R relation are shown in Figures 4.5 and 4.6, respectively.

We find that AP4 is consistent with the constraints for $-2\beta_{11} < 3$ cm. For lower values the maximum mass is below the observed maximum mass and for higher values the M-R relation does not pass through the contours of the constraint.

Again, we find that a new stable solution branch, for which $dM/d\rho_c > 0$ is satisfied, exists for low values of $\beta$.

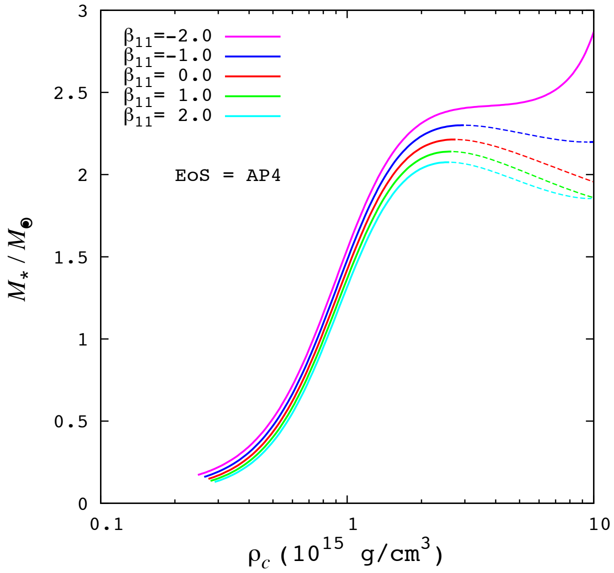


**Figure 4.5**: The $\rho_c - M$ relation for the AP4. The notation in the figure is the same as that of Figure 4.1 and the results are discussed in the text.

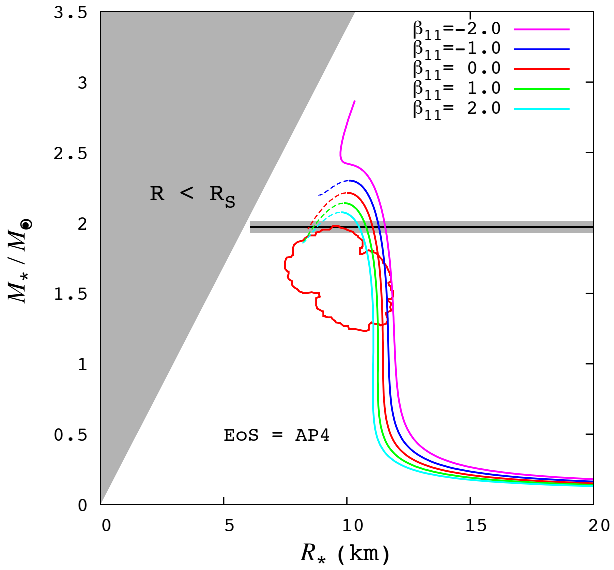


**Figure 4.6**: M-R relation for the AP4. The notation in the figure is the same as that of Figure 4.2 and the results are discussed in the text.

### 4.5.4 GS1

The central density versus the mass of the neutron star relation, $\rho_c - M$, and M-R relation are shown in Figures 4.7 and 4.8, respectively.

For GS1 the maximum mass in GR remains well below the measured mass of PSR J1614-2230. Starting from $\beta_{11} = -2$ the stability condition ($dM/d\rho_c > 0$) is satisfied for the whole range of central densities considered. For $\beta_{11} < -5$ the mass of the star can have very large values well exceeding the observed maximum mass of neutron stars.

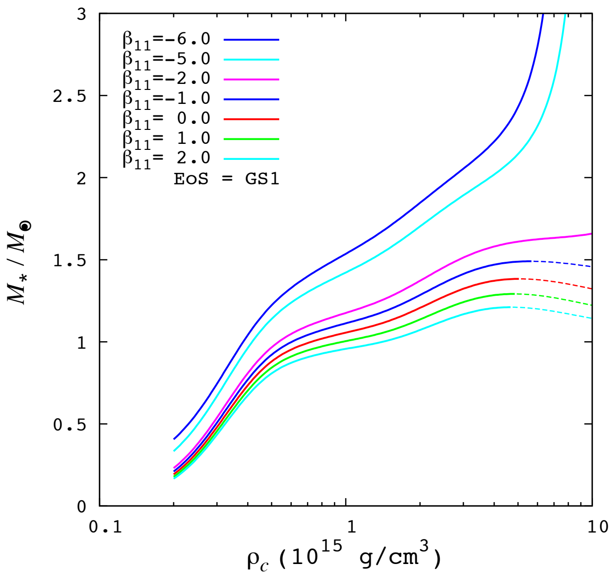


**Figure 4.7**: The $\rho_c - M$ relation for the GS1. The notation in the figure is the same as that of Figure 4.1 and the results are discussed in the text.

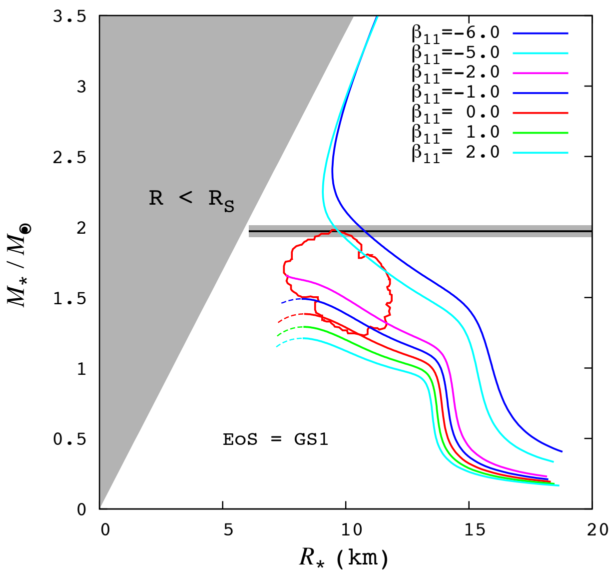


**Figure 4.8**: M-R relation for the GS1. The notation in the figure is the same as that of Figure 4.2 and the results are discussed in the text.

### 4.5.5 MPA1

The central density versus the mass of the neutron star relation, $\rho_c - M$, and M-R relation are shown in Figures 4.10 and 4.9, respectively.

MPA1 provides a maximum mass above the observed mass of PSR J1614-2230 in GR, though it does not pass through the M-R constraint of [1].

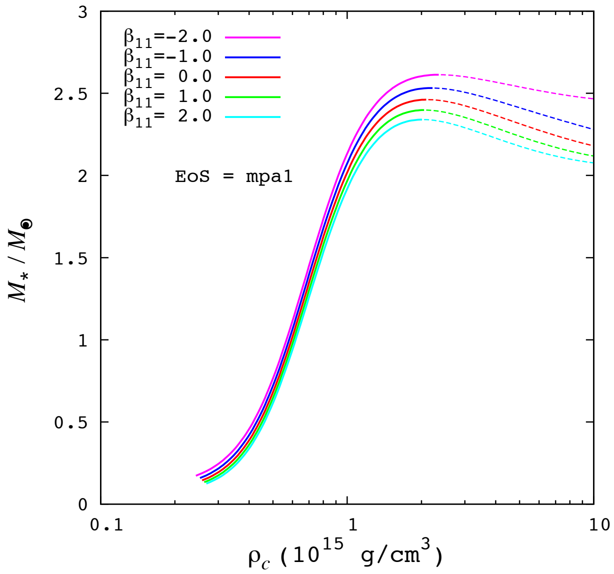


**Figure 4.9**: The $\rho_c - M$ relation for the MPA1. The notation in the figure is the same as that of Figure 4.1 and the results are discussed in the text.

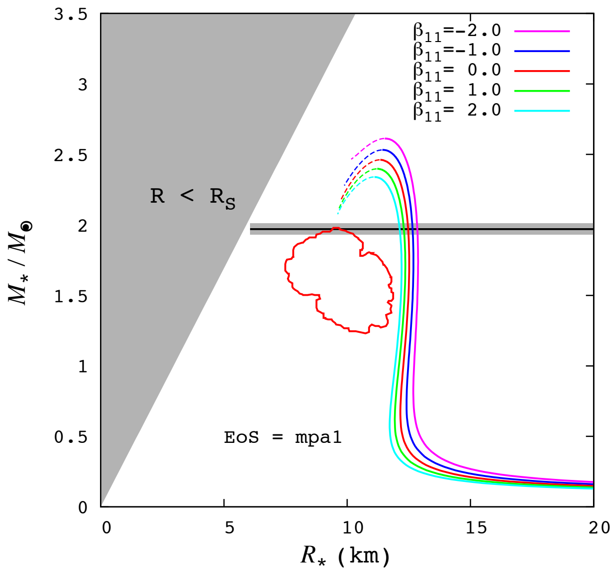


**Figure 4.10**: M-R relation for the MPA1. The notation in the figure is the same as that of Figure 4.2 and the results are discussed in the text.

### 4.5.6 MS1

The central density versus the mass of the neutron star relation, $\rho_c - M$, and M-R relation are shown in Figures 4.11 and 4.12, respectively.

The maximum mass for MS1 satisfies the observed mass of PSR J1614-2230 only for $\beta_{11} < -2$ though it moves away from the M-R constraint of [1] for such low values of $\beta$.

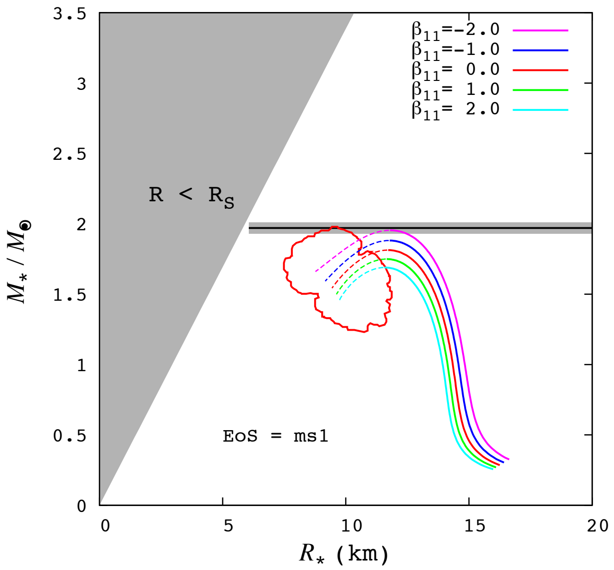


**Figure 4.11**: The $\rho_c - M$ relation for the MS1. The notation in the figure is the same as that of Figure 4.1 and the results are discussed in the text.

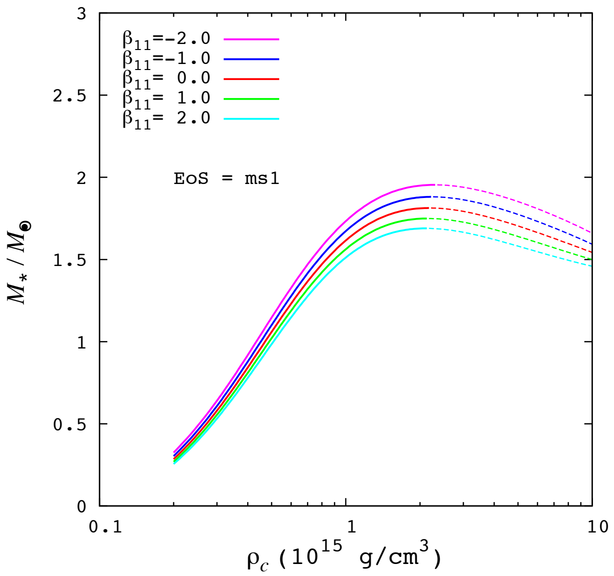


**Figure 4.12**: M-R relation for the MS1. The notation in the figure is the same as that of Figure 4.2 and the results are discussed in the text.

## 4.6 Dependence of Maximum Mass on $\beta$

For all EoS' we observe that the maximum stable mass of a neutron star, $M_{\rm max}$, and its radius at this mass, $R_{\rm min}$, increases for decreasing values of $\beta$, for the ranges we consider in the figures. There is no change in the behavior of $M_{\rm max}$ and $R_{\rm min}$ values while $\beta$ changes sign. Thus the structure of neutron stars in GR ($\beta = 0$) does not constitute an extremal configuration in terms of $M_{\rm max}$ and $R_{\rm min}$.

In Figure 4.13 we show the dependence of these quantities on the value of $\beta$ for GS1 EoS We fit the numerical results with cubic polynomials. For the maximum mass fitted with

$$\frac{M_{\rm max}}{M_\odot} = A\beta_{11}^3 + B\beta_{11}^2 + C\beta_{11} + M_0 \tag{4.22}$$

where $M_0$ is the maximum mass obtained for general relativity, we find that $A = -0.00132722 \pm 0.000118$, $B = 0.00864691 \pm 0.0001327$ and $C = -0.0893493 \pm 0.0002673$.

The upper and lower bounds on the value of $\beta$ presented for each EoS are in the range of $|\beta| \sim 10^{11}$ cm$^2$. Values of $|\beta|$, that are an order of magnitude smaller than this value, produce results that can not be distinguished from the results obtained within GR. This corresponds to a curvature scale of $R_0 \sim \beta^{-1} \sim 10^{-10}$ cm$^{-2}$ and a corresponding length scale of $L \sim \beta^{1/2} \sim 10^6$ cm, which is an order of magnitude smaller than the radius of the neutron star.

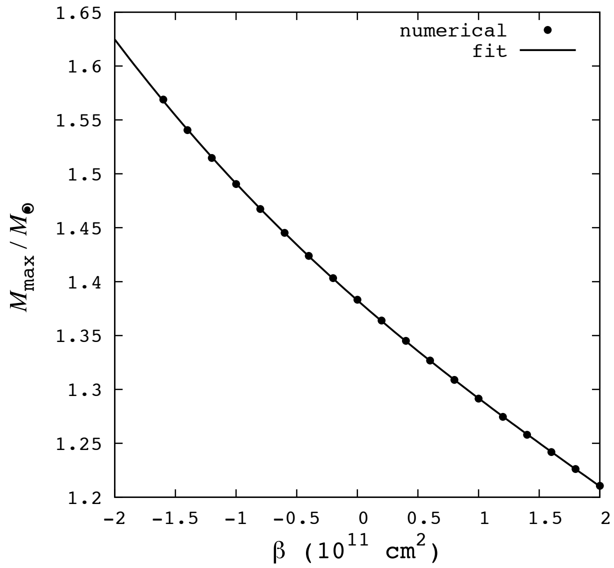


**Figure 4.13**: $M_{\rm max}$ changing with $\beta$ for the GS1 EoS.

# 5. CONCLUSIONS

In this thesis we studied the structure of neutron stars in an alternative theory of gravity motivated by string theory. In the first chapter we discussed the motivations for modifying gravity. In the second chapter we derived the field equations of this gravity model from its action. In Chapter 3 we obtained the hydrostatic equilibrium equations in spherical symmetry from the field equations by using a perturbative approach in which general relativity stands for the zeroth order gravity model. In Chapter 4 we have solved the hydrostatic equilibrium equations for neutron stars by using numerical methods. In order to solve the equations we have used realistic equations of state that describes the dense matter inside neutron stars and obtained the mass-radius relations depending on $\beta$, the free parameter of the modified gravity model considered. These mass-radius relations are then compared with recent observational constraints for neutron stars to constrain the value of $\beta$.

We have shown that observationally significant changes on the mass-radius relation are induced for $\beta$ exceeding $10^{11}$ cm$^2$. An order of magnitude smaller values for $\beta$ gives results that can not be differentiated from the results of general relativity. An order of magnitude greater values, on the other hand, leads to results that can not be associated with known properties of neutron stars. As these results hold for all equations of state we employed, we conclude from this analysis that $|\beta| \lesssim 10^{12}$ cm$^2$ is an observational constraint. This is a robust result for neutron stars obtained for all representative equations of state.

We find that, some of the equations of state, which do not give mass-radius relations consistent with the observations within the framework of general relativity, can be reconciled with these observations via the free parameter $\beta$ in the gravity model considered in this thesis. This then brings in the question of degeneracy between the equation of state and the free parameter $\beta$. This degeneracy does not effect the constraint we obtained for all equations of state. We show that the sole effect within the constrained values is to change the maximum mass of the neutron star.

We finally comment that the constraint we obtained is actually the strongest constraint we could obtain by using neutron stars. As these objects has radius $\sim 10$ km, the only length scale in the system, the corresponding radius of curvature is $10^{-12}$ cm$^{-2}$ and so variations on the mass-radius relation should eventually be obtained for $\beta \sim 10^{12}$ cm$^2$. As we obtain such variations in this limit we infer that the actual value of $\beta$ must be much smaller than this. As we mentioned before, deviations from general relativity are not significant for values much less than this value.

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## APPENDICES

**APPENDIX A:** Christoffel symbols and Riemann tensor
**APPENDIX B:** Calculation of D
**APPENDIX C:** Levi-Civita connection in terms of determinant of metric

**APPENDIX A: Christoffel Symbols and Riemann Tensor**

We start with the spherically symmetric metric tensor:

$$g_{mn} = \begin{bmatrix} e^{2\phi} & 0 & 0 & 0 \\ 0 & -e^{2\lambda} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^{-2}\theta \end{bmatrix}$$

Here $\nu$ and $\lambda$ are functions of radial coordinate and are independent of time. The definition of the Christoffel symbol is

$$\Gamma_{j,kn} = \frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x^n} + \frac{\partial g_{nj}}{\partial x^k} - \frac{\partial g_{kn}}{\partial x^j}\right)$$

All non-zero Christoffel symbols of the first kind for the metric are as follows,

$\Gamma_{0,01} = \Gamma_{0,10} = \frac{1}{2}\left(\partial_r g_{00} + \partial_0 g_{10} - \partial_0 g_{01}\right) = \phi' e^{2\phi}$

$\Gamma_{1,00} = \frac{1}{2}\left(\partial_0 g_{10} + \partial_0 g_{01} - \partial_r g_{00}\right) = -\phi' e^{2\phi}$

$\Gamma_{1,11} = \frac{1}{2}\left(\partial_r g_{11} + \partial_r g_{11} - \partial_r g_{11}\right) = -\lambda' e^{2\lambda}$

$\Gamma_{1,22} = \frac{1}{2}\left(\partial_\theta g_{12} + \partial_\theta g_{21} - \partial_r g_{22}\right) = r$

$\Gamma_{1,33} = \frac{1}{2}\left(\partial_\phi g_{13} + \partial_\phi g_{31} - \partial_r g_{33}\right) = r \sin^2\theta$

$\Gamma_{2,12} = \Gamma_{2,21} = \frac{1}{2}\left(\partial_\theta g_{21} + \partial_r g_{22} - \partial_\theta g_{12}\right) = -r$

$\Gamma_{2,33} = \frac{1}{2}\left(\partial_\phi g_{23} + \partial_\phi g_{32} - \partial_\theta g_{33}\right) = r^2 \cos\theta \sin\theta$

$\Gamma_{3,13} = \Gamma_{3,31} = \frac{1}{2}\left(\partial_\phi g_{31} + \partial_r g_{33} - \partial_\phi g_{13}\right) = -r \sin^2\theta$

$\Gamma_{3,23} = \Gamma_{3,32} = \frac{1}{2}\left(\partial_\phi g_{32} + \partial_\theta g_{33} - \partial_\phi g_{23}\right) = -r^2 \cos\theta \sin\theta$

We now list the Christoffel symbols of the second kind (Levi-Civita connections),

$$\Gamma^p_{kn} = g^{pj}\Gamma_{j,kn},$$

which are

$\Gamma^0_{01} = \Gamma^0_{10} = g^{0j}\Gamma_{j,00} = g^{00}\Gamma_{0,01} = \phi'$

$\Gamma^1_{00} = g^{1j}\Gamma_{j,00} = g^{11}\Gamma_{1,00} = \phi' e^{2(\phi-\lambda)}$

$\Gamma^1_{11} = g^{1j}\Gamma_{j,11} = g^{11}\Gamma_{1,11} = \lambda'$

$\Gamma^1_{22} = g^{1j}\Gamma_{j,22} = g^{11}\Gamma_{1,22} = -re^{-2\lambda}$

$\Gamma^1_{33} = g^{1j}\Gamma_{j,33} = g^{11}\Gamma_{1,33} = -r\sin^2\theta e^{-2\lambda}$

$\Gamma^2_{12} = \Gamma^2_{21} = g^{2j}\Gamma_{j,12} = g^{22}\Gamma_{2,12} = \frac{1}{r}$

$\Gamma^2_{33} = g^{2j}\Gamma_{j,33} = g^{22}\Gamma_{2,33} = -\sin\theta\cos\theta$

$\Gamma^3_{13} = \Gamma^3_{31} = g^{3j}\Gamma_{j,13} = g^{33}\Gamma_{3,13} = \frac{1}{r}$
$\Gamma^3_{23} = \Gamma^3_{32} = g^{3j}\Gamma_{j,23} = g^{33}\Gamma_{3,23} = \frac{\cos\theta}{\sin\theta} = \cot\theta$
Ricci tensors can be derived from these results using the formula:

$$R_{kn} = \partial_n \Gamma^j_{kj} - \partial_j \Gamma^j_{kn} + \Gamma^p_{kj}\Gamma^j_{pn} - \Gamma^p_{kn}\Gamma^j_{pj}$$

Therefore non-zero components of Ricci tensor are

$$\begin{aligned}
R_{00} &= -\partial_r \Gamma^1_{00} - \Gamma^1_{00}\Gamma^j_{1j} + 2\Gamma^1_{00}\Gamma^0_{10} \\
&= e^{2(\phi-\lambda)}\left(-\phi'' + \phi'\lambda' - \phi'^2 - \frac{2}{r}\phi'\right) \\
R_{11} &= \partial_r \Gamma^j_{1j} - \partial_r \Gamma^1_{11} - \Gamma^1_{11}\Gamma^j_{1j} + \left(\Gamma^0_{10}\right)^2 + \left(\Gamma^1_{11}\right)^2 + \left(\Gamma^2_{12}\right)^2 + \left(\Gamma^3_{13}\right)^2 \\
&= \phi'' - \phi'\lambda' + \phi'^2 - \frac{2}{r}\lambda' \\
R_{22} &= \partial_\theta \Gamma^j_{2j} - \partial_r \Gamma^1_{22} + 2\Gamma^1_{22}\Gamma^2_{12} + \left(\Gamma^3_{23}\right)^2 - \Gamma^1_{22}\Gamma^j_{1j} \\
&= e^{-2\lambda}\left(1 + r\phi' - r\lambda'\right) - 1 \\
R_{33} &= -\partial_r \Gamma^1_{33} - \partial_\theta \Gamma^2_{33} - \Gamma^1_{33}\Gamma^j_{1j} - \Gamma^2_{33}\Gamma^j_{2j} + 2\Gamma^1_{33}\Gamma^3_{13} + 2\Gamma^2_{33}\Gamma^3_{23} \\
&= \sin^2\theta\left[e^{-2\lambda}\left(1 + r\phi' - r\lambda'\right) - 1\right]
\end{aligned}$$

## APPENDIX B: Calculation of D

Here we calculate term $D$ of Chapter 2. There it is defined as (**2.66**)

$$\begin{aligned} D &= \int d^4x\sqrt{-g}\beta R^{\mu\nu} g_{\lambda\mu}\nabla_\rho\nabla_\nu\left(\delta g^{\lambda\rho}\right) \\ &= \int d^4x\sqrt{-g}\beta R^{\mu\nu}\nabla_\rho\nabla_\nu\left(g_{\lambda\mu}\delta g^{\lambda\rho}\right) \end{aligned} \quad \textbf{(B.1)}$$

Defining $C_{\nu\mu}{}^{\rho} = \nabla_\nu\left(g_{\lambda\mu}\delta g^{\lambda\rho}\right)$ we get

$$\begin{aligned} D &= \int d^4x\sqrt{-g}\beta R^{\mu\nu}\nabla_\rho C_{\nu\mu}{}^{\rho} \\ &= \beta\int d^4x\sqrt{-g}R^{\mu\nu}\left(\begin{array}{c} \partial_\rho C_{\nu\mu}{}^{\rho} - \Gamma^{\alpha}{}_{\rho\nu}C_{\alpha\mu}{}^{\rho} \\ -\Gamma^{\alpha}{}_{\rho\mu}C_{\nu\alpha}{}^{\rho} + \Gamma^{\rho}{}_{\rho\alpha}C_{\nu\mu}{}^{\alpha} \end{array}\right) \end{aligned} \quad \textbf{(B.2)}$$

Combining and arranging terms we obtain

$$D = -\beta\int d^4x\sqrt{-g}\left[\left(\partial_\rho R^{\mu\nu}\right) + \Gamma^{\nu}{}_{\rho\alpha}R^{\mu\alpha} + \Gamma^{\mu}{}_{\rho\alpha}R^{\alpha\nu}\right]C_{\nu\mu}{}^{\rho} \quad \textbf{(B.3)}$$

In the last equation we recognize the expression inside the brackets as $\nabla_\rho R^{\mu\nu}$. Writing explicit form of $C_{\nu\mu}{}^{\rho}$ we get

$$D = -\beta\int d^4x\sqrt{-g}\nabla_\rho R^{\mu\nu}\nabla_\nu\left(g_{\lambda\mu}\delta g^{\lambda\rho}\right) \quad \textbf{(B.4)}$$

Defining $D_\mu{}^{\rho} = g_{\lambda\mu}\delta g^{\lambda\rho}$ and $L_\rho{}^{\mu\nu} = \nabla_\rho R^{\mu\nu}$ we obtain

$$D = -\beta\int d^4x\sqrt{-g}L_\rho{}^{\mu\nu}\nabla_\nu D_\mu{}^{\rho} \quad \textbf{(B.5)}$$

Then,

$$D = -\beta\int d^4x\left[\begin{array}{c} -\left(\partial_\nu\sqrt{-g}\right)L_\rho{}^{\mu\nu}D_\mu{}^{\rho} - \left(\partial_\nu L_\rho{}^{\mu\nu}\right)\sqrt{-g}D_\mu{}^{\rho} \\ +\Gamma^{\rho}{}_{\nu\alpha}\sqrt{-g}L_\rho{}^{\mu\nu}D_\mu{}^{\alpha} - \Gamma^{\alpha}{}_{\nu\mu}\sqrt{-g}L_\rho{}^{\mu\nu}D_\alpha{}^{\rho} \end{array}\right] \quad \textbf{(B.6)}$$

Arranging terms we find

$$D = \beta\int d^4x\sqrt{-g}\left[\left(\partial_\nu L_\rho{}^{\mu\nu}\right) - \Gamma^{\alpha}{}_{\nu\rho}L_\alpha{}^{\mu\nu} + \Gamma^{\mu}{}_{\nu\alpha}L_\rho{}^{\alpha\nu} + \Gamma^{\nu}{}_{\nu\alpha}L_\rho{}^{\mu\alpha}\right]D_\mu{}^{\rho} \quad \textbf{(B.7)}$$

Expression inside the brackets is $\nabla_\nu L_\rho{}^{\mu\nu}$. Therefore we finally get

$$D = \beta\int d^4x\sqrt{-g}\nabla_\rho\nabla_\nu R^{\lambda\rho}g_{\mu\lambda}\delta g^{\mu\nu} \quad \textbf{(B.8)}$$

after inserting forms of $D_\mu{}^{\rho}$ and $L_\rho{}^{\mu\nu}$.

## APPENDIX C: Levi-Civita connection in terms of determinant of metric

The definition of the Christoffel symbol in terms of the partial derivatives of the metric tensor is

$$\begin{aligned}\Gamma^{\nu}{}_{\nu\lambda} &= \frac{1}{2}g^{\nu\sigma}\left(\partial_{\nu}g_{\sigma\lambda}+\partial_{\lambda}g_{\nu\sigma}-\partial_{\sigma}g_{\nu\lambda}\right)\\ &= \frac{1}{2}\left(\partial^{\sigma}g_{\sigma\lambda}+g^{\nu\sigma}\partial_{\lambda}g_{\nu\sigma}-\partial^{\nu}g_{\nu\lambda}\right)\end{aligned} \quad (\mathbf{C.1})$$

In the equation above, the first and the last terms are the same. Therefore, the Equation (**C.1**) will be

$$\Gamma^{\nu}{}_{\nu\lambda} = \frac{1}{2}g^{\nu\sigma}\partial_{\lambda}g_{\nu\sigma} \quad (\mathbf{C.2})$$

On the other hand, from the definition of the $g = \det g_{\nu\sigma}$, we obtain

$$\frac{\partial_{\lambda}g}{g} = \frac{\det\partial_{\lambda}g_{\nu\sigma}}{\det g_{\nu\sigma}} \quad (\mathbf{C.3})$$

By putting Equation (**C.3**) in Equation (**C.2**), we get

$$\Gamma^{\nu}{}_{\nu\lambda} = \frac{1}{2}\frac{\partial_{\lambda}g}{g} \quad (\mathbf{C.4})$$

The partial derivative of $\sqrt{-g}$ equals;

$$\partial_{\lambda}\sqrt{-g} = -\frac{1}{2}\frac{\partial_{\lambda}g}{\sqrt{-g}} \quad (\mathbf{C.5})$$

Therefore, it can be written as

$$\frac{\partial_{\lambda}\sqrt{-g}}{\sqrt{-g}} = \frac{1}{2}\frac{\partial_{\lambda}g}{g} \quad (\mathbf{C.6})$$

Finally, by using the calculation above, we obtain

$$\Gamma^{\nu}{}_{\nu\mu} = \frac{\partial_{\mu}\sqrt{-g}}{\sqrt{-g}}. \quad (\mathbf{C.7})$$

**CURRICULUM VITAE**

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